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**REVIEW OF RESEARCH
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AN EXPONENTIALLY FITTED QUADRATIC SPLINE DIFFERENCE SCHEME ON A NON-UNIFORM MESH

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Abstract

The uniformly convergent quadratic spline difference scheme is derived for the problem: $Ly = -\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $q(x) \ge q > 0$, $y(0) = \alpha_0$, $y(1) = \alpha_1$. A non-uniform mesh which provides for the location of a larger number of points in the boundary layers is used.

1. Introduced

An exponentially fitted quadric spline difference scheme for the problem:

$$(1)\qquad \begin{cases} Ly = -\varepsilon y'' + q(x)y = f(x), & 0 < x < 1, \\ y(0) = \alpha_0, \quad y(1) = \alpha_1, & q(x) \ge q > 0, \end{cases}$$

on a uniform mesh is derived in [3]. In this paper we shall generalize those result on a non-uniform mesh. Since spline difference schemes have the same order of accuracy and the same matrix structures on a uniform and

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a non-uniform grid, we wanted to transfer this characteristic on to an exponentially fitted quadratic spline difference scheme. But, there were some difficulties in determining the fitting factor. Namely, in [2] the fitting factor is determined from the requirement that the truncation error for the boundary layer functions should be equal to zero. Simultaneously, we obtain that the corresponding matrix has an iverse monotone form, and the obtained fitting factor approximates ε with the error $o(h^2)$. For this purpose, in the case of a non-uniform mesh, we must solve a system of 3 non-linear equations with 3 unknowns, which leads to very complicated and unuseful expressions.

So, we decided to put some conditions on the grid (natural for singular perturbation problems) in order to achieve a second order accuracy as on a uniform mesh.

2. Derivation of the scheme

Let us substitute differential equation (1) by the equation:

$$-\sigma(x)y'' + q(x)y = f(x)\,, \tag{2}$$

where $\sigma(x)$ is the fitting factor which will be determined subsequently. The approximate solution of equation (2) should be sought in the form of the quadratic spline $v(x) \in C^1\,[0,1]$:

$$v(x) = v_j + (x - x_j)v_j^{(1)} + \frac{(x - x_j)^2}{2}\,v_j^{(2)}\,,$$

$$x \in [x_j,\ x_{j+1}],\quad 0 = x_0 < x_1 < \ldots x_n < x_{n+1} = 1.$$

The points of collocation are points $x_{j+1/2} = x_j + h_j/2$, where $h_j = x_{j+1} - x_j$. Constants v_j, $v_j^{(1)}$ and $v_j^{(2)}$, $j=1(1)n$, are determined from the system of equations:

$$-\sigma_j v_j^{(2)} + q_{j+1/2}\left(v_j + \frac{h_j}{2}\,v_j^{(1)} + \frac{h_j^2}{8}\,v_j^{(2)}\right) = f_{j+1/2}$$

$$v_j = v_{j-1} + h_{j-1}v_{j-1}^{(1)} + \frac{h_{j-1}}{2}\,v_{j-1}^{(2)}$$

$$v_j^{(1)} = v_{j-1}^{(1)} + h_{j-1}v_{j-1}^{(1)}$$

By the elimination of $v_j^{(1)}$ and $v_j^{(2)}$ from the above equations, we get the difference scheme:

$$r^- v_{j-1} + r^c v_j + r^+ v_{j+1} = q^- f_{j-1/2} + q^+ f_{j+1/2} \tag{3}$$

$$r^- = -\frac{1}{b_{j-1}}\left[\frac{2\sigma_{j-1}}{h_{j-1}^2} - \frac{1}{4}q_{j-1/2}\right],$$

$$r^+ = -\frac{1}{b_j}\left[\frac{2\sigma_j}{h_j^2} - \frac{1}{4}q_{j+1/2}\right],$$

$$r^c = -r^- - r^+ + \frac{1}{b_j}q_{j+1/2} + \frac{1}{b_{j-1}}q_{j-1/2},$$

$$b_j = \frac{2\sigma_j}{h_j} + \frac{1}{4}q_{j+1/2}h_j,$$

$$q^- = \frac{1}{b_{j-1}}, \qquad q^+ = \frac{1}{b_j}.$$

In determining the fitting factor σ_j we use the following lemma:

Lemma 1. [1] *Let* $y \in C^4[0,1]$. *Let* $q'(0) = q'(1) = 0$. *Then the solution of problem (1) has the form:*

$$y(x) = u(x) + w(x) + g(x), \text{ where:}$$

$$u(x) = p_0 \exp(-x\sqrt{q(0)/\varepsilon},$$

$$w(x) = p_1 \exp(-(1-x)\sqrt{q(1)/\varepsilon},$$

p_0 *and* p_1 *are bounded functions of* ε *independent of* x *and:*

$$|g^{(j)}| \le M(1 + \varepsilon^{1-(1/2)j}), \quad j=0(1)4$$

M *is a constant independent of* ε.

The local truncation error of scheme (3) for an arbitrary smooth function φ has the form:

$$\tau_j(\varphi) = R_h\varphi_j - Q_h(L\varphi)_j \quad \text{where:}$$

$$R_h v_j = r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1},$$

$$Q_h f_j = q^- f_{j-1/2} + q^+ f_{j+1/2}.$$

Lemma 2. *Let, in scheme (3),* σ_{j-1} *be replaced by* σ_j^- *and* σ_j *by* σ_j^+, *where:*

$$\sigma_j^- = h_{j-1}^2 q_{j-1/2} S_j(q_{j-1/2})/8,$$

$$\sigma_j^+ = h_j^2 q_{j+1/2} S_j(q_{j+1/2})/8,$$

$$S_j(t) = 1 + \frac{4(h_j + h_{j-1})}{s_j(t)} ,$$

$$s_j(t) = h_j \exp(h_{j-1} \sqrt{t/\varepsilon}) - (h_j + h_{j-1}) +$$

$$h_{j-1} \exp(-h_j \sqrt{t/\varepsilon}) \quad \text{for} \quad j = 0(1)i_1 \quad \text{and}$$

$$s_j(t) = h_j \exp(-h_{j-1} \sqrt{t/\varepsilon}) - (h_j + h_{j-1}) +$$

$$h_{j-1} \exp(h_j \sqrt{t/\varepsilon}) \quad \text{for} \quad j = i_2(1)n+1, \quad i_1 \le i_2 .$$

For $i_1 < j < i_2$ *in the previous expressions we put* $h_j = h = \text{const.}$

Then:

a) $\tau_j(u) = 0, \quad j = 0(1)i_1$ *for* $q(x) = q = \text{const.}$

b) $\tau_j(w) = 0, \quad j = i_2(1)n+1$ *for* $q(x) = q = \text{const.}$

c) $\tau_j(u) = \tau_j(w) = 0$ *for* $i_1 < j < i_2$.

d) The matrix of system (3) *is an inverse monotone.*

Proof. a) By the direct substitution of σ_j and σ_{j-1} in the expression $R_h u_j$, we obtain zero. Since $Lu_j = 0$, we have that $\tau_j(u) = 0$.

In the same way we obtain that *b*) and *c*) hold.

d) Since $r^- \le 0$, $r^+ \le 0$, $r^c > (-r^- + r^+)$ the proof follows.

Throughout the paper M and δ denote different constants independent of ε and h_j.

Lemma 3. *Let:*

$$(4) \qquad \begin{cases} 0 \le h_j - h_{j-1} \le M_j \dfrac{h_{j-1}}{\varepsilon} \min(M_0 h_{j-1}^2, \varepsilon), & 1 \le j \le i_1 , \\ 0 \le h_{j-1} - h_j \le M_j \dfrac{h_j}{\varepsilon} \min(\bar{M}_0 h_j^2, \varepsilon), & i_2 \le j \le n+1 , \end{cases}$$

$h_j / h_{j\pm1} \le M, \quad h_j = h = \text{const.}, \quad i_1 < j < i_2 ,$

$M_i \le M, \quad i = 0(1)n+1, \quad \bar{M}_0 < M.$

Then:

$$\max(|\sigma_j^- - \varepsilon|, \ |\sigma_j^+ - \varepsilon|) \le M h_j^2, \quad j = 1(1)n+1.$$

Proof. Let $\varepsilon \le M_0 h_j^2$. Then:

$$\left| \frac{h_j + h_{j-1}}{s_j(t)} \right| \le M \exp(-h_{j-1} \sqrt{t/\varepsilon}) , \qquad |S_j(t)| \le M ,$$

$$\max(|\sigma_j^+|,\ |\sigma_j^-|) \le Mh_j^2 \quad \text{and} \quad |\sigma_j^\pm - \varepsilon| \le Mh_j^2 .$$

Let $\varepsilon \ge M_0 h_j^2$. After some Taylor expansions we have:

$$\frac{4(h_j+h_{j-1})}{s_j(q_{j-1/2})} = \frac{8}{h_j h_{j-1} q_{j/1/2}} + 2\,\frac{h_j-h_{j-1}}{h_j h_{j-1}}\sqrt{\frac{\varepsilon}{q_{j-1/2}}} + N , \quad .$$

$|N| \le M$, and from (4) we obtain $|\sigma_j^- - \varepsilon| \le Mh_j^2$. In the same manner we obtain $|\sigma_j^+ - \varepsilon| \le Mh_j^2$.

3. Proof of uniform convergence

Lemma 4. *Let* $y(x) \in C^4[0,1]$ and $q'(0) = q'(1) = 0$. *Let* σ_j *and* σ_{j-1} *in* (3) *be determined by Lemma 2. Let Lemma 3 hold. Then:*

$$|\tau_j(y)| \le \begin{cases} M\dfrac{h_j^3}{\varepsilon}, & M_0 h_{j-1}^2 \le \varepsilon \ \text{ or } \ h_j^2 \bar{M}_0 \le \varepsilon , \\[2ex] Mh_j, & M_0 h_j^2 \ge \varepsilon \ \text{ or } \ h_j^2 \bar{M}_0 \ge \varepsilon , \end{cases}$$

Proof. According to Lemma 1 we have:

$$\tau_j(y) = \tau_j(u) + \tau_j(w) + \tau_j(g) .$$

We shall estimate separately the truncation error for each function u, w, and g. We shall start with $u = u(x)$. Denote by $\tilde{\tau}_j(u)$ the truncation error $\tau_j(u)$ in the case $q(x) = q_0 = q(0) = \text{const}$. Since $\tilde{\tau}_j(u) = 0$, we have that:

(5) $$\tau_j(u) = \tau_j(u) - \tilde{\tau}_j(u), \quad j = 0(1)i_2-1.$$

After some Taylor developments we have:

(6) $$\tau_j(y) = T_2 y_j'' + T_3 y_j''' + \frac{r^- h_{j-1}^4}{4!}\, y^{IV}(\xi_1) + \frac{r^+ h_j^4}{4!}\, y^{IV}(\xi_2) +$$
$$\varepsilon\left[q^+ \frac{h_j^2}{8}\, y^{IV}(\xi_3) - q^- \frac{h_{j-1}^2}{8}\, y^{IV}(\xi_4)\right] + q^- \frac{h_{j-1}^4}{384}\, y^{IV}(\xi_5) -$$
$$q^+ \frac{h_j^4}{384}\, y^{IV}(\xi_6), \quad x_{j-1} \le \xi_i \le x_{j+1}, \quad i = 1(1)6.$$

$$T_2 = r^- \frac{h_{j-1}^2}{2} + r^+ \frac{h_j^2}{2} + \varepsilon\,(q^+ + q^-) - q^- \frac{h_{j-1}^2}{8} - q^+ \frac{h_j^2}{8} ,$$

$$T_3 = - \frac{h_{j-1}^3}{3!} r^- + \frac{h_j^3}{3!} r^+ + \varepsilon(-q^- \frac{h_{j-1}}{2} + q^+ \frac{h_j}{2}) + q^- q_{j-1/2} \frac{h_{j-1}^3}{48} -$$

$$q^+ q_{j+1/2} \frac{h_j^3}{48} . \text{ Let } h_{j-1} M_0 \le \varepsilon .$$

Taking into account (6) and (5), we obtain that:

$$|\tau_j(u)| \le M \frac{q(x_j)-q(0)}{\varepsilon^2} h_j^3 u_j \le M \frac{h_j^3}{\varepsilon} , \quad j = 1(1)i_2-1.$$

According to Lemma 1, from (6) and (4) we have:

$$\max\,(|\tau_j(w)|,\ |\tau_j(g)|) \le Mh_j^3/\varepsilon , \quad j = 1(1)i_2-1.$$

For $i_2 \le j \le n+1$ and $\bar{M}_0 h_j^2 \le \varepsilon$ we have:

$$|\tau_j(w)| = |\tau_1(w) - \tilde{\tau}_j(w)| \le M \frac{q(x_j)-q(0)}{\varepsilon^2} h_j^3 w_j \le Mh_j^3/\varepsilon ,$$

$\max(|\tau_j(u)|, |\tau_j(g)| \le Mh_j^3/\varepsilon$. Thus, Lemma 4 holds for $h_{j-1}M_0 \le \varepsilon$, or $h_j\bar{M}_0 \le \varepsilon$.

In the opposite case we use a truncation error in the form:

$$\tau_j(y) = r^- \frac{h_{j-1}^2}{2!} y''(\xi_1) + \frac{r^+ h_j^2}{2!} y''(\xi_2) + \varepsilon(q^- y''_{j-1/2} + \tag{7}$$

$$q^+ y''_{j+1/2}) - q^- \frac{h_{j-1}^2}{8} y''(\xi_3) - q^+ \frac{h_j^2}{8} y''(\xi_4) ,$$

$$x_{j-1} \le \xi_i \le x_{j+1} , \quad i = 1(1)4.$$

Since $|r^\mp| \le Mh_j^{-1}$ and $|q^\mp| \le M_j^{-1}$, we have that:

$$\max\,(|\tau_j(w)|,\ |\tau_j(u)|) \le Mh_j , \quad j = 1(1)i_2-1,$$

and from $\tau_j(u) = \tau_j(u)-\tilde{\tau}_j(u)$ we obtain that the same holds for function u.

For $i_2 \le j \le n+1$ we have that $\tau_j(w) = \tau_j(w) - \tilde{\tau}_j(w)$ and from (7) we can conclude that:

$\max(|\tau_j(w)|,\ |\tau_j(u)|,\ |\tau_j(g)|) \le Mh_j$. Thus, Lemma 4 holds.

Theorem 1. *Let $y(x) \in C^4\,[0,1]$. Let $q'(0) = q'(1) = 0$. Let the mesh points satisfy condition (4). Let v_j, $j = 0(1)n+1$, be the approximation to the solution $y(x_j)$ obtained by using (3).*

Then:

$|y(x_j)-v_j| \le Mh_j^2$, $j=0(1)n+1$, *where* M *is a constant independent of* ε *and* h_j.

Proof. Since:

$$r^- + r^c + r^+ \ge \begin{cases} M\varepsilon/h_j, & \text{for } M_0h_{j-1} \le \varepsilon \text{ or } \bar{M}_0h_j \le \varepsilon \\ Mh_j, & \text{for } M_0h_{j-1} \ge \varepsilon \text{ or } \bar{M}_0h_j \ge \varepsilon, \end{cases}$$

from the inequalities:

$R_h[\pm(y(x_j) - v_j) + Mh^{-2}] \ge 0$, $\bar{h} = \max(h_j, h_{j-1}) = \text{const.}$ we obtain the statement of Theorem 1 (the inverse monotony of the corresponding matrix and Lemma 4).

Remark 1. For $h_j = h = \text{const}$, $j = 0(1)n$, scheme (3) reduces to the one given in [3].

For $\sigma_j = \varepsilon = 1$ scheme (3) reduces to the one corresponding to the collocation spline given in [2].

5. Numerical results

Scheme (3) was used to obtain the approximate solution of problem [1]:

$$-\varepsilon y'' + (1 + x(1-x))y = f(x),$$

$$y(0) = y(1) = 0.$$

Its exact solution is:

$$y(x) = 1 - (1-x)\exp(-x/\sqrt{\varepsilon}) - x\exp((x-1)/\sqrt{\varepsilon}).$$

Here, we shall present the numerical results which suggest the choice of the grid.

Tables 1, 2 and 3 contain the maximal differences between the exact and approximate solution at the points of the grid, for different ε and n.

Tabel 1 contains the results on an equidistant grid.

Table 1.

$$\max_j |v_j - v(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.17-04	0.42-05	0.10-05	0.26-06	0.66-07
1/4	0.29-04	0.73-05	0.18-05	0.43-06	0.11-06
1/8	0.40-04	0.10-04	0.25-05	0.62-06	0.16-06
1/16	0.43-04	0.11-04	0.27-05	0.67-06	0.17-06
1/32	0.49-04	0.12-04	0.31-05	0.77-06	0.19-06
1/64	0.66-04	0.16-04	0.41-05	0.10-05	0.26-06
1/128	0.93-04	0.23-04	0.57-05	0.14-05	0.36-06
1/256	0.12-03	0.33-04	0.81-05	0.20-05	0.51-06
1/512	0.19-03	0.46-04	0.11-04	0.24-05	0.72-06
1/1024	0.26-03	0.63-04	0.16-04	0.41-05	0.10-05

A non-equidistant grid is formed in such a way that more of points are found in layers than outside them.

On the interval $[0, c_0]$, c_0 is a constant which is given in advance, the grid is non-equidistant and obtained according to the formula:

$$\tilde{h}_j = \tilde{h}_{j-1} + M\,\frac{\tilde{h}_{j-1}}{\varepsilon}\,\min(\tilde{h}^2_{j-1},\ \varepsilon), \qquad j = 1(1)n_1-1,$$

$$h_j = Q\tilde{h}_j, \quad Q = c_0/\tilde{Q}, \quad \tilde{Q} = \sum_{j=1}^{n_1-1} \tilde{h}_j, \quad x_j = x_{j-1} + h_{j-1}, \quad j = 1(1)n_1,$$

$x_0 = 0$, $\tilde{h}_0$, M and c_0 are given, $M_0 = \bar{M}_0 = Q^{-2}$.

On the interval $[c_0, 1-c_0]$, the grid is equidistant:

$h_j = (1-2c)/n_2$, $\quad j = n_1(1)n_1+n_2-1$,

$n+1 = 2n_1 + n_2$, $\quad x_j = x_{j-1} + h$, $\quad j = n_1+1,\ n_1+n_2-1$.

On the interval $[1-c_0, 1]$, the grid is symmetric to the grid on interval $[0, c_0]$. This is a starting mesh. In each next spet the intervals are halved.

Table 2

$c_0 = 0.05,\ h_0 = 0.01,\ M=46,\ n_1(n+1) = 3/16$

$$\max_j |v_j - y(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.21-04	0.54-05	0.14-05	0.34-06	0.86-07
1/4	0.59-04	0.15-04	0.37-05	0.94-06	0.24-06
1/8	0.99-04	0.25-04	0.64-05	0.16-05	0.40-06
1/16	0.15-03	0.39-04	0.99-05	0.25-05	0.63-06
1/32	0.25-03	0.67-04	0.17-04	0.44-05	0.11-05
1/64	0.38-03	0.10-03	0.26-04	0.67-05	0.17-05
1/128	0.50-03	0.13-03	0.35-04	0.89-05	0.22-05
1/256	0.57-03	0.15-03	0.38-04	0.96-05	0.24-05
1/512	0.52-03	0.12-03	0.31-04	0.79-05	0.20-05
1/1024	0.38-03	0.78-04	0.18-04	0.45-05	0.11-05
1/2048	0.28-03	0.77-04	0.15-04	0.50-05	0.13-05
1/216	0.43-03	0.25-03	0.11-03	0.41-04	0.13-04

(37% points in the layers)

Table 3

$c_0 = 0.05,\ h_0 = 0.01,\ M=7,5,\ n_1/(n+1) = 5/16$

$$\max_j |v_j - y(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.55-04	0.14-04	0.37-05	0.93-06	0.23-06
1/4	0.16-03	0.41-04	0.10-04	0.26-05	0.66-06
1/8	0.27-03	0.70-04	0.18-04	0.45-05	0.11-05
1/16	0.39-03	0.11-03	0.28-04	0.70-05	0.12-05
1/32	0.64-03	0.18-03	0.47-04	0.12-04	0.31-05
1/64	0.91-03	0.26-03	0.71-04	0.18-04	0.47-05
1/128	0.11-02	0.33-03	0.92-04	0.24-04	0.62-05
1/256	0.11-02	0.35-03	0.99-04	0.26-04	0.69-05
1/512	0.12-02	0.31-03	0.84-04	0.22-04	0.58-05
1/1024	0.78-03	0.18-03	0.44-04	0.11-04	0.29-05
1/216	0.23-03	0.75-04	0.22-04	0.64-05	0.18-05
1/219	0.64-04	0.22-04	0.71-05	0.36-05	0.14-05

(62,5% points in the layers)

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Rezime

EKSPONENCIJALNO FITOVANA SPLAJN DIFERENCNA ŠEMA SA KVADRATNIM SPLAJNOVIMA NA NERAVNOMERNOJ MREŽI

Izvedena je uniformno konvergentna splajn diferencna šema sa kvadratnim splajnovima za problem $Ly = -\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $q(x) \geq q > 0$, $y(0) = \alpha_0$, $y(1) = \alpha_1$. Korišćena je neravnomerna mreža koja omogućuje smeštanje većeg broja tačaka u granične slojeve.

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ON NUMERICAL SOLUTION OF A TURNING POINT PROBLEM

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Abstract

A numerical method for a singular perturbation problem with a turning point is considered. The method uses non-equidistant discretization meshes. The first order accuracy, uniform in the perturbation parameter is proved in the discrete L^1 norm.

1. Introduction

In this paper we consider a numerical method for the following singularly perturbed boundary value problem with a turning point:

$$\varepsilon^2 u'' + xb(u)u' = f_\varepsilon(x), \quad x \in I = [a, 1], \tag{1a}$$

$$u(a) = A, \quad u(1) = B \tag{1b}$$

with $a=0$ or $a=-1$. By ε we denote the perturbation parameter: $0 < \varepsilon \le 1$ (usually $\varepsilon \ll 1$). The functions b, f_ε and numbers A, B are given. Our basic assumptions are:

$$b,\ f_\varepsilon \in C^1(I), \tag{2a}$$

$$\beta := \min_{x \in I} b(x) > 0, \tag{2b}$$

$$|f_\varepsilon(x)| \le M(|x| + F_\varepsilon(x)), \ x \in I, \tag{2c}$$

$$|f'_\varepsilon(x)| \le M(1 + (\varepsilon^{-2}|x| + \varepsilon^{-1})\ F_\varepsilon(x)), \ \ x \in I, \tag{2d}$$

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where

$$F_\varepsilon(x) = exp(-\int_0^x c(t)dt/\varepsilon^2), \quad c(x) := xb(x).$$

Here and throughout the paper M denotes any positive constant, independent of ε. For simplicity, in the rest of the paper the dependence of f_ε and F_ε on ε will not be denoted. Furthermore, we require:

(3) $$b'(x) > -3\beta/2, \quad x \in I.$$

We shall begin our investigation with the case $a=0$, which will be considered in Section 2. In Section 3 we shall give necessary modifications concerning the case $a=-1$.

Section 2 contains several parts. In the first part we investigate problem (1) and we give estimates for the derivatives of its solution, u. We use the assumption (2) to prove the appropriate estimates. For instance, (2b,c) guarantee that u is bounded uniformly in ε. Note that (2c,d) allow the case $f\equiv0$, for which the asymptotic behaviour of the solution has usually been investigated, [8]. Since u has an $O(\varepsilon)$ boundary layer at $x=0$, a special numerical method should be applied. We shall use finite-differenc schemes on a special non-equidistant mesh which is dense in the layer. The mesh, I_h, (with the mesh points $0=x_0<x_1<\ldots<x_n=1$, $n \in N$), is generated by a suitable function λ (i.e. $x_i = \lambda(ih)$, $h=1/n$). λ depends on ε in such a way that the smaller ε becomes, the more the mesh is condensed in the layer. Essentially, λ is a certain modification of the inverse of the boundary layer function. The finite-difference discretization and its stability are discussed in the second part of Section 2. The stability is proved in the discrete L^1 norm. The assumption (3) is important for our stability result. In the third part of Section 2 we deal with the consistency error using the derivative estimates from the first part and properties of function λ. In this part constants M are independent of h as well. Our basic result is

(4) $$\|w_h - u_h\|_h \le h.$$

Here and throughout the paper we use the following notation:

$w_h = [w_1, w_2, \ldots, w_{n-1}]^T \in R^{n-1}$ is the numerical solution,

$u_h = [u(x_1), u(x_2), \ldots, u(x_{n-1})]^T \in R^{n-1}$ is the discretization of the continuous solution,

$$\|z_h\|_h = \sum_{i=1}^{n-1} \bar{h}_i |z_i|, \quad z_h = [z_1, z_2, \dots z_{n-1}]^T \in R^{n-1},$$

$\bar{h}_i = (h_i + h_{i+1})/2$, $h_i = x_i - x_{i-1}$. We can think of (4) as a first order convergences uniform in ε (which is the final aim of all numerical methods for singular perturbation problems) even though $\|\cdot\|_h$ depends on ε since h_i's depend on it. Still, $\|\cdot\|_h$ is the usual discrete L^1 norm on the non--equidistant mesh, cf. [1]. We end Section 2 by presenting some numerical results.

Up till now, numerical methods for linear turning point problems have usually been considered in the case when the left hand side of (1a) contains the additional term $d(x)u$, where $d(x)>0$, $x \in I$, or at least $d(0)>0$, see [3], [4], [5], [7]. In the first three of these papers equidistant discretizations only are considered and upwind or exponentially fitted schemes are used. Paper [7] uses mesh generation - the approach which we shall apply here. This approach dates from 1969, [2], where a self-adjoint singularly perturbed boundary value problem was considered, and it has been modified and applied to other types of problems with a small parameter, see [7], [9], [10], [11], for instance. In [12], the authors deal with problems of type (1), among others, investigating the ill conditioning of the corresponding exponentially fitted discretization on the equidistant mesh.

2. Case a=0.

Throughout this Section we shall consider problem (1) with a=0, assuming that (2) holds.

2.1 Analysis of the Continuous Problem

For the proof of the following lemma we do not need the assumption (2d):

Lemma 1. *Problem (1) has a unique solution* $u \in C^3(I)$ *which is bounded uniformly in* ε:

$$|u(x)| \le M, \ x \in I.$$

Proof. Consider the operators

$$Lu := -\varepsilon^2 u'' - xb(x)u', \quad Ru := (u(0), u(1)). \tag{5}$$

Since (L, R) is inverse monotone, uniqueness of the solution is guaranteed. Existence and boundedness uniform in ε follow because there exist unifomly bounded upper and lower solutions to the problem (1). To show that use

$$v(x) = M_1(2-x) + M_2F(x)$$

with appropriate constants M_1, M_2 (independent of ε), such that $v(0) \ge |A|$, $v(1) \ge |B|$ and

$$Lv(x) = M_1 xb(x) + M_2c'(x)\ F(x) \ge |f(x)|. \tag{6}$$

The inequality in (6) can be achieved because of (2b,c). To see this, note that there exists a number $\delta \in (0,1]$ (independent of ε), such that $c'(x) \ge \gamma > 0$ for $x \in [0,\delta]$. Then, if $0 \le x \le \delta$ use

$$Lv(x) \ge M_1\beta x + M_2\gamma F(x) \ge |f(x)|.$$

On the other hand, if $\delta \le x \le 1$:

$$Lv(x) \ge M_1\beta x + M_2c'(x)\ F(x) \ge |f(x)|.$$

Hence, the upper solution is $v(x)$ and the lowe one is $-v(x)$. □

Lemma 2. $|u^{(i)}(0)| \le M\varepsilon^{-i}$, $i=1,2$.

Proof. In the case $i=2$, the proof follows from (1a). For $i=1$ rewrite (1a) in the form

$$\varepsilon^2u'' + (cu)' = f + c'u$$

and integrate this equation from 0 to x_*, where $x_* \in (0,\varepsilon)$ is such a point that $u'(x_*) = (u(\varepsilon) - u(0))/\varepsilon$ (hence $|u'(x_*)| \le M/\varepsilon$). □

Lemma 3. $|u'(x)| \le M(1 + (x\varepsilon^{-2} + \varepsilon^{-1})\ F(x))$, $x \in I$.

Proof. For the technique cf. [6]. Rewrite (1a) as follows:

$$\varepsilon^2(u'/F)' = f/F$$

and express u' by integration. Then because of (2c) and Lemma 2 we have:

$$|u'(x)| \le M(S + \varepsilon^{-1}F(x)),$$

where

$$S = \varepsilon^{-2}\int_0^x (t + F(t))\ (F(x)/F(t))dt.$$

Since

$$\int_x^t sb(s)ds \le \beta(t^2-x^2)/2 \ , \ 0 \le t \le x,$$

it follows

$$S \le x\varepsilon^{-2}F(x) + \varepsilon^{-2}\int_0^x t \ exp(0.5\beta(t^2-x^2)/\varepsilon^2)dt \le x\varepsilon^{-2}F(x) + M \ . \ \square$$

Lemma 4. $|xu''(x)| \le M(1 + \varepsilon^{-1}exp(-\beta x^2/(4\varepsilon^2)))$, $x \in I$.

Proof. Differentiate (1a) and obtain

$$\varepsilon^2(u''/F)' = (f'-c'u')/F \ .$$

Then because of (2d) and Lemmas 2 and 3:

$$|u''(x)| \le M(\varepsilon^{-2}\int_0^x(F(x)/F(t))dt + (x^2\varepsilon^{-4}+ x\varepsilon^{-3})F(x) + \varepsilon^{-2}F(x)) \ .$$

Since

$$x\varepsilon^{-2}\int_0^x (F(x)/F(t))dt \le x\varepsilon^{-2}\int_0^x exp(0.5\beta(t-x)x/\varepsilon^2)dt \le M \ ,$$

and

$$x^k\varepsilon^{-(k+1)}F(x) \le M\varepsilon^{-1}exp(-\beta x^2/(4\varepsilon^2)) \ , \quad k=1,2,3 \ ,$$

the lemma is proved. □

Using previous lemmas, we can obtain the following estimates which will be used in 2.3 in the analysis of the consistency error.

Theorem 1. *For $x \in I$ we have*

(7a) $|(cu)''(x)| \le M(1+\varepsilon^{-1}y(x))$,

(7b) $\varepsilon^2|u'''(x)| \le M(1+\varepsilon^{-1}y(x))$,

(7c) $\varepsilon^2|u''(x)| \le M(|x| + y(x))$,

(7d) $|(cu)'(x)| \le M$,

where $y(x) = exp(-|x|/\varepsilon)$.

Proof. Note that for any $\sigma>0$ independent of ε it holds that

$$exp(-\sigma x^2/\varepsilon^2) \le M\, y(x) .$$

Then the inequalities (7a) and (7d) follow easily from Lemmas 3 and 4, while (7c) follows directly from (1a). Finally, differentiate (1a) and use Lemmas 3 and 4 to obtain (7b). □

2.2 The Discretization and Its Stability

Note that

$$\|z_h\|_h = \|Hz_h\|_1$$

where $H= diag(\bar{h}_1,\ \bar{h}_2, \dots\ \bar{h}_{n-1})\in R^{n-1,n-1}$ and $\|\cdot\|_1$ is the usual vector norm in R^{n-1}. The corresponding matrix norm is:

$$\|A_h\|_h = \|HA_hH^{-1}\|_1 ,$$

where A_h is a matrix in $R^{n-1,n-1}$ and $\|\cdot\|_1$ is the usual matrix norm in that space.

The discrete operator on the mesh I_h is:

$$L_hz_i := -\varepsilon^2D''_hz_i - D'_hc(x_i)z_i + c'(x_i)z_i$$

and it corresponds to L from (5) which is rewritten in the conservative form

$$Lu = -\varepsilon^2u'' - (cu)' + c'u .$$

Here

$$D''_hz_i = (h_{i+1}z_{i-1} - 2\bar{h}_iz_i + h_iz_{i+1})/(h_ih_{i+1}\bar{h}_i) ,$$

and

$$D'_hz_i = (z_{i+1}-z_i)/\bar{h}_i ,$$

cf. [1]. Thus, the discrete problem reads:

$$(8)\qquad \begin{gathered} L_hw_i = -f(x_i) ,\ i=1,2,\dots n-1 ,\\ w_0=A ,\ w_n=B . \end{gathered}$$

Rewrite (8) in the matrix form:

$$(9)\qquad A_hw_h = d_h ,$$

where $d_h= [d_1,d_2,\dots d_{n-1}]^T\in R^{n-1}$, $d_i=-f(x_i)$, $i=2,3,\dots n-2$,
$d_1=-f(x_1) + \varepsilon^2A/(h_1\bar{h}_1)$, $d_{n-1}=-f(x_{n-1}) + \varepsilon^2B/(h_n\bar{h}_{n-1}) + c(x_n)B/\bar{h}_{n-1}$,
and A_h is the corresponding tridiagonal matrix.

Now we shall prove stability of the discrete problem (9) in the norm $\|\cdot\|_h$.

Theorem 2. *Let* $b \in C^1(I)$, $c'(x) \geq \gamma > 0$, $x \in I$, *and let* (2b) *hold. Then:*

(10) $$\|A_h^{-1}\|_h \leq M \ .$$

Proof. Because of the upwind scheme it is easy to see that A_h is an *L*-matrix (the diagonal elements are positive and the off diagonal elements are non-negative.) Furthermore:

$$(HA_hH^{-1})^T e_h \geq \gamma e_h \ ,$$

where $e_h = [1,1,\ldots,1]^T \in R^{n-1}$. Thus A_h is an M-matrix ($A_h^{-1} \geq 0$) and the result follows. □

However, we can show that the first condition of Theorem 2 can be weakened and replaced by (3), provided that $h=1/n$ be sufficiently small (but independent of ε). For that purpose we have to specify our discretization mesh. Let us define the mesh generating function λ:

(11) $$\lambda(t) = \begin{cases} \omega(t) := Q\varepsilon t/(q-t) \ , & t \in [0,\alpha] \\ \pi(t) \ , & t \in [\alpha,1] \end{cases} \ .$$

Here $\alpha \in (0,1)$ is an arbitrary parameter (independent of ε),

(12) $$q = \alpha + \varepsilon^{1/3} \ ,$$

and $\pi(t)$ is a third order polynomial, such that $\lambda \in C^2(I)$ and $\pi(1)=1$. The parameter Q should be positive and chosen in such a way that $\pi''' \geq 0$ (a simple analysis shows that $Q \in (0,1/2]$ would suffice for all $\varepsilon \in (0,1]$ and $\alpha \in (0,1)$). This implies

$$\pi^{(k)}(t) \geq \pi^{(k)}(\alpha) = \omega^{(k)}(\alpha) > 0 \ , \ t \in [\alpha,1],$$

firstly for $k=2$ and then for $k=1$. Obviously:

$$\omega^{(k)}(t) > 0 \ , \ k=0,1,\ldots, \ t \in [0,\alpha] \ ,$$

and taking (12) into account:

(13a) $$0 < \lambda^{(k)}(t) \leq M \ , \ k=1,2 \ , \ t \in I.$$

Furthermore, note the inequality:

(13b) $$exp(-\omega(t)/\varepsilon) \leq M \, exp(-M/(q-t)) \ , \ t \in (0,q) \ ,$$

which will be used in Section 2.3.

The mesh points are given by

(14) $$x_i = \lambda(t_i)\ ,\ \ t_i = ih\ ,\ \ i=0,1,\dots,n\ .$$

There are other possible choices of function λ, cf. [9], but because of simplicity, (1) is the only form which we shall consider here.

Theorem 3. *Let $b \in C^1(I)$ and let (2b) and (3) hold. Then the discretization matrix A_h on the mesh (14), (11), with sufficiently small h and Q, independent of ε, satisfies (10).*

Proof. Let $e_h = [1+x_1,\ 1+x_2, \dots,\ 1+x_{n-1}]^T \in R^{n-1}$. Then for $i=1,2,\dots,\ n-1$ we have:

$$((HA_hH^{-1})^T e_h)_i \ge (1+x_i)c'(x_i) + (h_i/\bar{h}_i)c(x_i) =: s_i\ .$$

Let δ and γ have the same meaning as in the proof of Lemma 1.

Then if $x_i \le \delta$:

$$s_i \ge \gamma > 0\ .$$

Now let $x_i > \delta$. We shall prove

(15) $$s_i \ge \gamma_1 > 0\ ,$$

for some γ_1 independent of ε and h, and the proof will be completed. Rewrite s_i in the form:

$$s_i = s_i' - p_i\ ,$$

where

$$s_i' = (1+x_i)c'(x_i) + c(x_i)\ ,$$

$$p_i = ((h_{i+1}-h_i)/(h_{i+1}+h_i))\ c(x_i) > 0\ .$$

Note that by (3) there exists a constant $\mu > 0$ (independent of ε), such that $b'(x) \ge \mu - 3\beta/2$. Then:

$$s_i' \ge \mu(1+x_i)x_i + (\beta/2)(-3x_i^2+x_i+2) > \mu(1+\delta)\delta\ .$$

Next we shall show that

(16) $$p_i \le Mh$$

and (15) will follow provided that h be sufficiently small. Let us first note that

$$\pi(t) = a_3(t-\alpha)^3 + a_2(t-\alpha)^2 + a_1(t-\alpha) + a_0\ ,$$

where

$$a_0=\omega(\alpha)=Q\alpha\varepsilon^{2/3} \ , \quad a_1=\omega'(\alpha)=Qq\varepsilon^{1/3} \ , \quad a_2=\omega''(\alpha)/2=Qq \ ,$$
$$a_3=(1-Q(q(1-\alpha)^2-q(1-\alpha)\varepsilon^{1/3}-\alpha\varepsilon^{2/3}))/(1-\alpha)^3 \ .$$

Then it is easy to see that there exist sufficiently small parameter Q and sufficiently small constant $m > 0$ (both independent of ε), such that

$$\pi(\alpha+m) = \lambda(\alpha+m) \le \delta < m_1 = \lambda(t_1) \ .$$

This implies $t_i > \alpha+m$ and $t_{i-1} \ge \alpha+m$, if h is small enough. Then because of (13a) we have

$$p_i \le M\, h^2\lambda''(t_{i+1})/(2h\lambda'(t_{i-1})) \ ,$$

and (16) follows from (13a) and $\lambda'(t_{i-1}) \ge \lambda'(\alpha+m) > 2Qqm$. □

Remark. The requirement in Theorem 3 that the parameter Q should be sufficiently small is neeeded to make the coefficient $a_0=Q\alpha\,\varepsilon^{2/3}$ sufficiently small. However, if ε is small enough, there is no need for such a constraint on the parameter Q.

2.3 The Convergence Result

Let us consider the consistency error

$$r_i := L_h u(x_i)-(Lu)(x_i) \ , \quad i=1,2,\dots,n-1 \ .$$

We have

$$r_i = r_i'' + r_i' \ ,$$

$$r_i'' = \varepsilon^2(u''(x_i)-D_h''u(x_i)) \ , \quad r_i' = g'(x_i)-D_h'g(x_i) \ ,$$

where $g(x) := c(x)u(x)$. Let $r_h = [r_1, r_2, \dots, r_{n-1}]^T \in R^{n-1}$. Then we have:

Theorem 4. *Let* (2) *hold. Let discrete operator* L_h *be given on the mesh* (14), (11). *Then* $\|r_h\|_h \le Mh$.

Proof. We shall prove

$$|r_i''| \le Mh \ , \tag{17a}$$

$$h_{i+1}|r_i'| \le Mh^2 \ , \tag{17b}$$

and the result will follow. The following estimates hold:

$$|r_i''| \le M\varepsilon^2 h_{i+1}|u'''(\sigma_i'')| \ , \quad \sigma_i'' \in (x_{i-1},x_{i+1}) \ , \tag{18a}$$

$$|r_i''| \le 2\varepsilon^2 \max_{x_{i-1}\le x \le x_{i+1}} |u''(x)| \ ,$$

$$|r_i'| \le ((h_{i+1}-h_i)/(h_i+h_{i+1}))\,|g'(\sigma_i')| + G_i/h_{i+1} \,,$$
$$\sigma_i' \in (x_i, x_{i+1}) \,,$$

where

$$(19) \qquad G_i = \int_{x_i}^{x_{i+1}} (x_{i+1}-x)\,|g''(x)|\,dx \,.$$

To prove (17b) it is sufficient to show

$$(20) \qquad G_i \le Mh^2 \,,$$

since from (7d) and (13a) we have

$$h_{i+1}((h_{i+1}-h_i)/(h_i+h_{i+1}))\,|g'(\sigma_i')| \le M(h_{i+1}-h_i) \le Mh^2 \,.$$

Let us prove (17a) and (20). We shall use the technique which is essentially the one from [2], cf. [7], [9], [10], [11]. The proof is devided into three steps:

1° $t_{i-1} \ge \alpha - \varepsilon^{1/3}$,

2° $t_{i-1} \le \alpha - 3h$,

3° $\alpha - 3h < t_{i-1} < \alpha - \varepsilon^{1/3}$

1° From (18a), (13) and (7b) we have:

$$|r_i''| \le Mh(1+\varepsilon^{-1}\, y(x_{i-1})) \le Mh(1+\varepsilon^{-1} y(\lambda(\alpha - \varepsilon^{1/3}))) \le Mh \,.$$

Similarly, from (19), (7a) and (13) it follows

$$G_i \le Mh^2(1+\varepsilon^{-1}\, y(x_i)) \le Mh^2 \,.$$

2° In this case we have $\alpha - t_{i+1} \ge (\alpha - t_{i-1})/3$ and thus $q - t_{i+1} > (q - t_{i-1})/3$. Now (18a), (13) and (7b) give

$$|r_i''| \le Mh(1+(q-t_{i+1})^{-2}\, y(\lambda(t_{i-1}))) \le Mh \,.$$

In the same way:

$$G_i \le Mh^2(1+(q-t_{i+1})^{-4}\, y(\lambda(t_i))) \le Mh^2 \,.$$

3° Now $\varepsilon^{1/3} < 3h$. We use (18b) and (7c) to obtain

$$|r_i''| \le M(\lambda(t_{i+1}) + y(\lambda(t_{i-1}))) \le M(\lambda(\alpha) + h\lambda'(t_{i+1}) + y(\lambda(\alpha - 3h))) \le Mh.$$

On the other hand

$$G_i \le M \int_{x_i}^{x_{i+1}} (x_{i+1}-x)(1+\varepsilon^{-1} y(x))\,dx ,$$

and after integration:

$$G_i \le M(h^2+ hy(x_i)) \le M(h^2+ hy(\lambda(\alpha-3h))) \le Mh^2 . \ \square$$

Thus, combining Theorems 2 and 3 with Theorem 4, we can get the corresponding convergence results:

Theorem 5. *Let u be the solution to the continuous problem* (1) *and let* (2) *and* $c'(x)\ge\gamma>0$, $x \in I$, *hold. Let* w_h *be the solution to the discrete problem* (8) *on the mesh* (14), (11). *Then we have:*

$$\|w_h-u_h\|_h \le Mh . \tag{21}$$

Theorem 6. *Let u be the solution to the continuous problem* (1) *and let* (2) *and* (3) *hold. Let* w_h *be the solution to the discrete problem* (8) *on the mesh* (14), (11) *with sufficiently small h and Q, independent of ε. Then* (21) *holds.*

2.4 Numerical Results

We shall present results concerning the following test problem:

$$-\varepsilon^2 u'' - 2xu' = 2exp(-(x/\varepsilon))^2 , \ u(0)=1 , \ u(1)= exp(-1/\varepsilon^2) ,$$

whose solution is known: $u(x) = exp(-(x/\varepsilon)^2)$. Table 1 contains the results obtained by using the mesh generating function (11) with paremeters Q=1, α=0.5 . By changing these parameters we can change the density of the mesh in the layer. In this case the percentage of the mesh points which lie within the layer varies from about 25% to about 35% (it changes slightly when ε and n change).

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TABLE 1.

ε	n	50	100	200
1.-2	E	1.47-2	7.54-3	3.82-3
	E_h	2.08-4	1.06-4	5.40-5
1.-3	E	1.72-2	8.94-3	4.54-3
	E_h	2.46-5	1.26-5	6.42-6
1.-4	E	1.88-2	9.79-3	4.98-3
	E_h	2.67-6	1.38-6	7.04-7

We use the following notation:

$$E=\|w_h-u_h\|_\infty\ ,\quad E_h=\|w_h-u_h\|_h\ ,$$

and, as usual, 1.-2 means 10^{-2}etc.

Table 1 shows more than our theory gives: the first order pointwise convergence uniform in ε can be observed, and the error E_h decreases when ε does. However, we can't expect this in general. Some other problems, which we have tested, show the first order L^1 convergence uniform in ε only, which coincide with our theory.

3. The Case a=-1

Now $I=[-1,1]$. First we shall show that Theorem 1 holds in this case as well. The only difference is in the proof of Lemma 1, where the function $v(x)$ should take the form:

$$v(x)=M_1(2-p(x))\ +\ M_2F(x)\ ,$$

where $p(x)$ is a $C^2(I)$-function, having the following properties: $p(x)=|x|$, $x\in I\setminus[-\delta,\delta]$; $p''(x)>0$, $xp'(x)\ge 0$, $x\in(-\delta,\delta)$.

Next, the discretization mesh (14) should be used with

$t_i=-1+ih,\ i=0,1,\dots,n\ ,\ h=2/n\ ,\ n=2k\ ,\ k\in\mathrm{N}$.

The mesh generating function coincides on [0,1] with λ from (11), and for $t\in[-1,0]$ we take $\lambda(t)=-\lambda(-t)$. Thus, λ'' is discontinuous at t=0, but this will not affect the proof of Theorem 4 since the two mesh steps in vicinity of x=0 are equidistant. At the mesh points x_i , $i=1,2,\dots,k-1$, the scheme L_h should be used with

$$D'_h z_i = (z_i - z_{i-1})/\bar{h}_i \quad .$$

Finally, Theorem 2 can be proved analogously, thus Theorem 5 follows too.

Remark. This paper contains the results of the first stage of investigations of the problems of type (1). We expect to improve these results in forthcoming papers.

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Rezime

O NUMERIČKOM REŠAVANJU PROBLEMA SA POVRATNOM TAČKOM

Posmatra se numerički metod za singularni perturbacioni problem sa povratnom tačkom. Metod koristi neekvidistantne mreže disktetizacije. U diskretnoj normi L^1 je dokazan prvi red tačnosti, uniforman po perturbacionom parametru.

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A NUMERICAL SOLUTION OF THE SINGULAR PERTURBATION PROBLEM ARISING FROM A WEAKLY COUPLED SYSTEM

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Abstract

A first order differential equation with a small perturbation parameter is derived from a weakly coupled first order system. It is solved numerically by the finite-difference method on a special discretization mesh.

1. Introduction

Let us consider the following system of first order differential equations:

$$y' = f(x) \ , \tag{1.a}$$

$$\varepsilon u' + a(x,u) = 0, \quad x \in I = [0,1] \ , \tag{1.b}$$

$$y(0) + b_0 u(0) = c_0 \ , \tag{1.c}$$

$$y(1) + b_1 u(1) = c_1 \ , \tag{1.d}$$

where ε is a small perturbation parameter, $0 < \varepsilon << 1$; $b_i, c_i \in R$, $i = 0,1$, $b_0 b_1 < 0$; f and a are sufficiently smooth functions in I and $I \times R$, respectively, and $a_u(x,u) > a_* > 0$, $x \in I$, $u \in R$.

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The physical meaning of problem (1) can be found in [7], where the linear case was considered. An exponentially fitted higher order scheme was introduced there and the convergence uniform in ε was proved. Here we propose a simpler approach which decouples y and u from (1) in the following way.

Let $g(x)$ be the solution to (1.a) satisfying the condition $g(0)=0$. Then we have

(2) $$y(x) = g(x) + d ,$$

where $d = y(0)$. Then the boundary conditions (1.c,d) reduce to

(3) $$b_0 u(0) = c_0 - d ,$$

$$b_1 u(1) = c_1 - g(1) - d .$$

By substracting these equations we eliminate d and get:

(4) $$u(0) + bu(1) = c ,$$

with the appropriate $b > 0$ and c. It is easy to evaluate $g(1)$ and thus to get c. Then problem (1.b), (4) can be solved numerically. By using the numerical approximation of $u(0)$ we can get d from (3) and find $y(x)$ from (2).

Thus we can consider problems of the type (1.b), (4) only. We shall solve them numerically using the classical finite-difference schemes on special discretization meshes. By the standard technique we can prove the first order convergence uniform in ε (see [1,2,8,9,10] for instance).

The solution to problem (1.b), (4) has a layer of width $O(\varepsilon)$ at the origin. Our discretization mesh is generated by a suitable function which maps an equidistant mesh to the mesh which is dense in the layer. The density changes automatically when ε does. Because of that we obtain uniform numerical results within the region where the continuous solution changes abruptly.

We note that the numerical methods for more general systems have been considered in a number of papers, let us mention [6] and [11] only. However, because of the special structure of system (1) we can construct here a very simple and efficient method.

Throughout the paper we denote by M any positive constant which is independent of ε and of the discretization mesh.

2. The Continuous Problem

Thus, we consider the problem:

$$(5)\qquad \begin{aligned} Tu &:= \varepsilon u' + a(x,u) = 0\ , \quad x \in I\ ,\\ B_b u &:= u(0) + bu(1) = c\ , \end{aligned}$$

$0 < \varepsilon \le \varepsilon_0$, $b \ge 0$, $a \in C^1(I \times R)$, $a_u(x,u) > a_* > 0$, $(x,u) \in I \times R$.

Let us denote by $u_b \in C^2(I)$ the solution to problem (5).

The case $b=0$ is well known. Operator (T, B_0) is inverse monotone and the solution u_0 exists uniquely. Using the technique from [1,8] (cf. [2] as well) we can prove (the details will be omitted):

Theorem 1. *For the solution* u_0 *to problem* (5) *with* $b=0$, *we have*

$$|u_0'(x)| \le M(1 + \varepsilon^{-1} v(x))\ ,$$

$$|u_0''(x)| \le M(\varepsilon^{-1} + \varepsilon^{-2} v(x)),$$

where $x \in I$ *and* $v(x) = \exp(-a_* x/\varepsilon)$.

Now we can prove

Theorem 2. *The solution* u_b *to problem* (5) *with* $b > 0$ *uniquely exists and its derivatives satisfy the same estimates as the derivatives of* u_0 *in Theorem* 1.

Proof. Let $z \in C^1(I)$ denote the unique solution to the reduced problem

$$a(x,z) = 0$$

and let

$$|z(x)| \le Z\ , \quad x \in I\ .$$

Then, by the inverse monotonicity of (T, B_0) we can get that the solution u_0 to problem

$$Tu = 0\ , \quad B_0 u = s,$$

satisfies

$$P := \min(s, -Z) \le u_0 \le \max(s, Z) =: Q\ .$$

Let us consider two values of s :

$$s_1 < \min(0, c-bZ)\ ,$$

$$s_2 > \max(0, c+bZ)\ .$$

We have the corresponding solutions $u_{0,i}$ and constants P_i, Q_i, $i=1,2$. Thus we get

$$Tu_{0,i} = 0, \quad i=1,2 ,$$

$$B_b u_{0,1} \le s_1 + bQ_1 = s_1 + bZ < c$$

and similarly

$$B_b u_{0,2} > c .$$

Because of the fact that u_0 depends continuously on s, we can conclude that there exists a unique value s_*, such that the solution $u_{0,*}$ to problem

$$Tu = 0 , \quad B_0 u = s_* ,$$

satisfies

$$B_b u_{0,*} = c .$$

Hence $u_{0,*} = u_b$ and we have the same estimates as in Theorem 1. The Theorem is proved.

We can see that u_b has a layer of width $O(\varepsilon)$ at $x=0$. The estimates of the derivatives of u_b are important for the proof of the convergence uniform in ε.

3. The Discretization

Let us use mesh I_h from [9], cf. [1,2,8,10]:

$$x_i = m(ih) , \quad i=0,1,\dots,n , \quad h=1/n , \quad n \in \mathbb{N} ,$$

$$m(t) = \begin{cases} F(t) := A\varepsilon t/(q-t) , & t \in [0,p] \\ F(p) + F'(p)(t-p) , & t \in [p,1] \end{cases}$$

Here $q \in (0,1)$ and $A \in (0,q/\varepsilon_0)$ are fixed numbers and $p \in (0,q)$ is the abscissa of the contact point of tangent line from $(1,1)$ to the curve $F(t)$. The point p can be found explicitly.

We form the discretization of problem (5) using finite difference scheme on mesh I_h:

(6a) $$w_0 + bw_n = c ,$$

(6b) $$T_h w_i := \varepsilon D_- w_i + a(x_i, w_i) = 0 , \quad i=1,2,\dots,n ,$$

where $\{w_i\}$ denotes a mesh function on I_h and

$$D_- w_i = (w_i - w_{i-1})/h_i ,$$

$$h_i = x_i - x_{i-1} \;, \quad i=1,2,\ldots,n \;.$$

The existence and uniqueness of the solution to the discrete problem (3) can be established in the same way as in the continuous case.

Theorem 3. *Let* $\{w_i\}$ *be the solution to the discrete problem* (6) *on the mesh* I_h *with* $n > 2/q$ *and let* u_b *be the solution to the continuous problem* (5). *Then we have*

$$|w_i - u_b(x_i)| \le M\,h \;.$$

Proof. By using the technique from [8,1] we can easily prove the consistency uniform in ε, i.e.:

$$|T_h u_b(x_i) - T_h w_i| =$$

$$|T_h u_b(x_i) - (Tu_b)(x_i)| =$$

$$\varepsilon\,|D_- u_b(x_i) - u'(x_i)| \le M\,h \;, \quad i=1,2,\ldots,n \;.$$

There remains to prove the stability uniform in ε. Let us first consider the linear case, i.e. problem (5) when $a(x,u) = a(x)u - r(x)$, $a(x) > a_* > 0$, $x \in I$. Its discretization on mesh I_h reads:

$$w_0 + bw_n = c \;,$$

$$w_i = A_i w_{i-1} + B_i \;, \quad i=1,2,\ldots,n \;,$$

where

$$A_i = \varepsilon/(\varepsilon + a(x_i)h_i) \;, \quad B_i = A_i r(x_i) h_i/\varepsilon \;.$$

We shall prove the stability inequality:

$$|w_i| \le M(|c| + R) \;, \tag{7}$$

where $|r(x_i)| \le R$, $i=1,2,\ldots n$. For the technique cf. [4]. We have

$$w_i = K_i w_0 + L_i \;, \quad i=1,2,\ldots,n \;,$$

$$K_i = A_i K_{i-1} \;, \quad i=1,2,\ldots,n \;, \; K_0 = 1 \;,$$

$$L_i = A_i L_{i-1} + B_i \;, \quad i=1,2,\ldots,n \;, \; L_0 = 0 \;,$$

$$w_0 = (c - bL_n)/(1 - bK_n) \;.$$

Since $0 < K_i < 1$, $i=1,2,\ldots,n$, inequality (7) follows if we show

$$|L_i| \le M\,R \;, \quad i=1,2,\ldots,n \;. \tag{8}$$

It holds that

(9) $$|L_i| \le L'_i \ , \quad i=1,2,\dots,n \ ,$$

where

$$L'_i = A'_i L'_{i-1} + B'_i \ , \quad i=1,2,\dots,n \ , \quad L'_0 = 0 \ ,$$
$$A'_i = \varepsilon/(\varepsilon + a_{\bullet} h_i) \ge A_i \ ,$$
$$B'_i = A'_i R h_i/\varepsilon \ge |B_i| \ .$$

By induction we can prove :

(10) $$L'_i \le R/a_{\bullet} \ , \quad i=1,2,\dots,n \ ,$$

and because of (9) we get (8), and (7) as well. Indeed, it is easy to verify that (10) holds for $i=1$. Now suppose that (10) is valid for some i. We have

$$L'_{i+1} \le A'_i R/a_{\bullet} + B'_i =$$
$$= A'_i R(1/a_{\bullet} + h_i/\varepsilon) = R/a_{\bullet} \ .$$

In the nonlinear case we have

$$T_h u_b(x_i) - T_h w_i =$$
$$\varepsilon D_-(u_b(x_i) - w_i) + Q_i(u_b(x_i) - w_i) \ ,$$

where

$$Q_i = \int_0^1 a_u(x_i, w_i + s(u_b(x_i) - w_i))ds > a_{\bullet} > 0 \ .$$

Hence, the linear convergence uniform in ε follows in the same way as in the linear case. The Theorem is proved.

4. Numerical Examples

We shall consider two problems from [3], modified to the form of our problem (5). The first problem is linear:

(11) $$\varepsilon u' + u = r(x) + \varepsilon r'(x) \ , \quad u(0) + u(1) = c \ ,$$

where

$$r(x) = 10 - (10 + x)\exp(-x) \ ,$$
$$c = r(1) + 10(1 + \exp(-1/\varepsilon)) \ ,$$

so that the solution reads $u = r(x) + 10\exp(-x/\varepsilon)$.

The second problem is

$$\varepsilon u' + u^2 = 0\,,$$

(12)

$$u(0) + u(1) = 10 + 1/(0.1 + 1/\varepsilon)\ .$$

Its solution reads $u = 1/(0.1 + x/\varepsilon)$.

In all the numerical experiments we use the mesh generating function m with parameters A=1, q=0.9, thus obtaining about 45 % of the mesh points in the interval $[0,\varepsilon]$ which represents the layer.

Let us denote by E the maximum pointwise error. We have achieved the same values of E for all ε's which were considered: $\varepsilon = 10^{-3}$, 10^{-6}, 10^{-9}. In the case of problem (11), we have E=0.126 for n=50 and E=0.0642 for n=100. In the case of problem (12), we have E=0.199 for n=100 and E=0.102 for n=200.

The nonlinear system (6) was solved by the Newton-Kantorovich method (see [5] for instance). The initial approximation was a constant mesh-function $c/(1 + b)$. The iterations were carried on until the maximal pointwise difference between two succesive iterations became less than 0.01. Nine iterations were needed.

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Rezime

NUMERIČKO REŠAVANJE SINGULARNO PERTURBOVANOG PROBLEMA KOJI PROIZILAZI IZ JEDNOG SLABO POVEZANOG SISTEMA

Diferencijalna jednačina prvog reda sa malim parametrom je izvedena iz jednog slabo povezanog sistema prvog reda. Zatim je rešavana numerički pomoću metoda konačnih razlika na specijalnoj mreži diskretizacije.

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AFFINE CONNECTIONS IN THE FINSLER SPACE

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ABSTRACT

In the Finsler space some general connection coefficients are introduced which depend on seven arbitrarily chosen parameters. Special cases of these are Cartan connection coefficients and some of the recurrent Finsler spaces. All of them have the property that the angle between two vectors by parallel displacement remains unchanged. It is shown that there are 2^7 essentially different types of connection coefficients of this kind. For the general case some fundamental relations are obtained.

§1. CONNECTION COEFFICIENTS IN D RECURRENT FINSLER SPACES

Let the metric function in the Finsler space F_N be denoted by $L(x,\dot{x})$, and the metric tensor by $g_{\alpha\beta}(x,\dot{x})$. Then,

$$(1.1)\qquad g_{\alpha\beta}(x,\dot{x}) = \dot{\partial}_\alpha \dot{\partial}_\beta F(x,\dot{x}), \qquad F(x,\dot{x}) = 2^{-1}L^2(x,\dot{x}).$$

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$L(x,\dot{x})$ is homogeneous of degree one in $\dot{x}$, $g_{\alpha\beta}(x,\dot{x})$ and any other tensor or vector field is supposed to be homogeneous of degree zero in $\dot{x}$.

If we denote by D the absolute differential which corresponds to the change of line element from $(x,\dot{x})$ to $(x+dx,\dot{x}+d\dot{x})$, then for arbitrary tensor field $T^{\alpha}_{\beta}(x,\dot{x})$ we define

$$DT^{\alpha}_{\beta}=dT^{\alpha}_{\beta}+(\Gamma^{*\alpha}_{\iota\gamma}T^{\iota}_{\beta}-\Gamma^{*\iota}_{\beta\gamma}T^{\alpha}_{\iota})dx^{\gamma}+(A^{\alpha}_{\iota\gamma}T^{\iota}_{\beta}-A^{\iota}_{\beta\gamma}T^{\alpha}_{\iota})\bar{D}\ell^{\gamma}, \tag{1.2}$$

where

$$\ell^{\gamma} = \frac{\dot{x}^{\gamma}}{L(x,\dot{x})} \tag{1.3}$$

$$\bar{D}\ell^{\gamma} = d\ell^{\gamma}+L^{-1}N^{\gamma}_{\beta}dx^{\beta}+M^{\gamma}_{\beta}\bar{D}^{\beta}. \tag{1.4}$$

N^{γ}_{β} is the coefficient of non-linear connection homogeneous of degree one in $\dot{x}$ and M^{γ}_{β} is a tensor. With respect to the coordinate transformation

$$x^{\alpha'}= x^{\alpha'}(x^{1},x^{2},\ldots,x^{N}) \Leftrightarrow x^{\alpha}=x^{\alpha}(x^{1'},x^{2'},\ldots,x^{N'}) \tag{1.5}$$

$$\alpha' = 1',2',\ldots,N' \qquad \alpha = 1,2,\ldots,N,$$

where both families of functions are sufficiently times differentiable and where

$$\dot{x}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\dot{x}^{\alpha} \quad \Leftrightarrow \quad \dot{x}^{\alpha} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\dot{x}^{\alpha'} \tag{1.6}$$

$$\frac{\partial}{\partial \dot{x}^{\theta'}} = \frac{\partial x^{\theta}}{\partial x^{\theta'}}\frac{\partial}{\partial \dot{x}^{\theta}} \quad , \frac{\partial}{\partial x^{\theta'}} = \frac{\partial x^{\theta}}{\partial x^{\theta'}}\frac{\partial}{\partial x^{\theta}} + \frac{\partial^{2}x^{\theta}}{\partial x^{\theta'}\partial x^{\alpha'}}\dot{x}^{\alpha'}\frac{\partial}{\partial \dot{x}^{\theta}}, \tag{1.7}$$

we have

$$N^{\alpha'}_{\gamma'} = \frac{\partial^{2}x^{\alpha}}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\dot{x}^{\beta'} + N^{\alpha}_{\gamma}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}, \tag{1.8}$$

$$M^{\alpha'}_{\gamma'} = M^{\alpha}_{\gamma}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}. \tag{1.9}$$

From (1.2) we have

$$(1.10)\qquad Dg_{\alpha\beta} = dg_{\alpha\beta}-(\Gamma^{*\delta}_{\alpha\gamma}\, g_{\delta\beta}+\Gamma^{*\delta}_{\beta\gamma}g_{\alpha\delta})dx^\gamma-(A^\delta_{\alpha\gamma}g_{\delta\beta}+A^\delta_{\beta\gamma}g_{\alpha\delta})\bar{D}\ell^\gamma.$$

Using (1.4) and (1.10) we get

$$(1.11)\qquad Dg_{\alpha\beta} = g_{\alpha\beta|\gamma}dx^\gamma+g_{\alpha\beta}\big|_\gamma\bar{D}\ell^\gamma,$$

where

$$(1.12)\qquad g_{\alpha\beta|\gamma} = \partial_\gamma g_{\alpha\beta}-\dot{\partial}_\delta g_{\alpha\beta}N^\delta_\gamma-\Gamma^\star_{\alpha\beta\gamma}-\Gamma^\star_{\beta\alpha\gamma}$$

$$(1.13)\qquad g_{\alpha\beta}\big|_\gamma = L(\dot{\partial}_\delta g_{\alpha\beta})(\delta^\delta_\gamma-M^\delta_\gamma) - A_{\alpha\beta\gamma}-A_{\beta\alpha\gamma}.$$

Definition 1.1. The Finsler space is D recurrent if there exist vector fields $\lambda_\gamma=\lambda_\gamma(x,\dot{x})$ and $\mu_\gamma=\mu_\gamma(x,\dot{x})$ such that

$$(1.14)\qquad Dg_{\alpha\beta} = K(x,\dot{x},dx,\bar{D}\ell)g_{\alpha\beta}\ ,$$

where

$$(1.15)\qquad K(x,\dot{x},dx,\bar{D}\ell) = \lambda_\gamma dx^\gamma+\mu^\gamma\bar{D}\ell^\gamma\ .$$

Such a Finsler space we shall denote by $F_N(D)$.

Theorem 1.1. In the space $F_N(D)$ in which $\lambda_\gamma,\mu_\gamma,N^\gamma_\beta,M^\gamma_\beta$ are given and they satisfy (1.4) and (1.10)-(1.15) the connection coefficients $\Gamma^{*\beta}_{\alpha\gamma}$, $A^\beta_{\alpha\gamma}$ depend on arbitrarily chosen tensors $\theta_{\alpha\beta}$, $\tilde{\Gamma}^\beta_{\alpha\gamma}$ and $\tilde{A}^\beta_{\alpha\gamma}$, where

$$(1.16)\qquad \begin{aligned}&(a)\quad \theta_{\alpha\beta} = \theta_{\beta\alpha}\\ &(b)\quad \tilde{\Gamma}^\beta_{\alpha\gamma} = \Gamma^{*\beta}_{\alpha\gamma}-\Gamma^{*\beta}_{\gamma\alpha}\\ &(c)\quad \tilde{A}^\beta_{\alpha\gamma} = A^\beta_{\alpha\gamma}-A^\beta_{\gamma\alpha}\end{aligned}$$

and have the form

$$
\begin{aligned}
\Gamma^{*}_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma} &-2^{-1}(N^{\delta}_{\gamma}\dot\partial_{\delta}g_{\alpha\beta}+N^{\delta}_{\alpha}\dot\partial_{\delta}g_{\beta\gamma}-N^{\delta}_{\beta}\dot\partial_{\delta}g_{\gamma\alpha}) \\
&-2^{-1}[\lambda_{\gamma}(g_{\alpha\beta}+\theta_{\alpha\beta})+\lambda_{\alpha}(g_{\beta\gamma}+\theta_{\beta\gamma})-\lambda_{\beta}(g_{\gamma\alpha}+\theta_{\gamma\alpha})] +2^{-1}(\tilde{\Gamma}_{\alpha\beta\gamma}-\tilde{\Gamma}_{\beta\gamma\alpha}+\tilde{\Gamma}_{\gamma\alpha\beta}) \\
&-[\theta_{\alpha\beta}(\Gamma^{*o}_{o\gamma}-L^{-1}N^{o}_{\gamma})+\theta_{\beta\gamma}(\Gamma^{*o}_{o\alpha}-L^{-1}N^{o}_{\alpha})-\theta_{\gamma\alpha}(\Gamma^{*o}_{o\beta}-L^{-1}N^{o}_{\beta})]
\end{aligned}
\tag{1.17}
$$

($\gamma_{\alpha\beta\gamma}$ is the Christoffel symbol),

$$
\begin{aligned}
A_{\alpha\beta\gamma} = 2^{-1}L\ &[(\delta^{\delta}_{\gamma}-M^{\delta}_{\gamma})\dot\partial_{\delta}g_{\alpha\beta}+(\delta^{\delta}_{\alpha}-M^{\delta}_{\alpha})\dot\partial_{\delta}g_{\beta\gamma})-(\delta^{\delta}_{\beta}-M^{\delta}_{\beta})\dot\partial_{\delta}g_{\gamma\alpha}] \\
&-2^{-1}(\mu_{\gamma}g_{\alpha\beta}+\mu_{\alpha}g_{\beta\gamma}-\mu_{\beta}g_{\alpha\gamma}) +2^{-1}(\tilde{A}_{\alpha\beta\gamma}-\tilde{A}_{\beta\gamma\alpha}+\tilde{A}_{\gamma\alpha\beta}) \\
&-2^{-1}[(\mu_{\gamma}+2\ell_{\gamma})\theta_{\alpha\beta}+(\mu_{\alpha}+2\ell_{\alpha})\theta_{\beta\gamma}-(\mu_{\beta}+2\ell_{\beta})\theta_{\alpha\gamma}] \\
&-[\theta_{\alpha\beta}(A^{o}_{o\gamma}-M^{o}_{\gamma})+\theta_{\beta\gamma}(A^{o}_{o\alpha}-M^{o}_{\alpha})-\theta_{\gamma\alpha}(A^{o}_{o\beta}-M^{o}_{\beta})].
\end{aligned}
\tag{1.18}
$$

Proof. From the relation $g_{\alpha\beta}\ell^{\alpha}\ell^{\beta} = 1$, using (1.14) and

$$D\ell^{\alpha} = d\ell^{\gamma}+\Gamma^{*\gamma}_{o\beta}dx^{\beta}+A^{\gamma}_{o\beta}\bar{D}\ell^{\gamma} \tag{1.19}$$

("o" means the contraction by ℓ) we obtain

$$[\lambda_{\gamma}+2\ell_{\kappa}(\Gamma^{*\kappa}_{o\gamma}-L^{-1}N^{\kappa}_{\gamma})]dx^{\gamma}+[\mu_{\gamma}+2\ell_{\gamma}+2\ell_{\kappa}(A^{\kappa}_{o\gamma}-M^{\kappa}_{\gamma})]\bar{D}\ell^{\gamma} = 0. \tag{1.20}$$

This is the crucial relation which shows that dx^{γ} and $\bar{D}\ell^{\gamma}$ are not independent. For $\lambda_{\gamma}=0$, $\mu_{\gamma}=0$, $\Gamma^{*\alpha}_{o\gamma}=L^{-1}N^{\alpha}_{\gamma}$, $A^{\alpha}_{o\gamma}=M^{\alpha}_{\gamma}$ (1.20) reduces to the well known formula $\ell_{\alpha}D\ell^{\alpha}=0$ in the non-recurrent Finsler space. If we multiply (1.20) with $\theta_{\alpha\beta}(x,\dot{x})$, $(\theta_{\alpha\beta}=\theta_{\beta\alpha})$ and add to the right hand side of (1.14) using (1.11), we have

$$
\begin{aligned}
g_{\alpha\beta|\gamma}dx^{\gamma}+g_{\alpha\beta}|_{\gamma}\bar{D}\ell^{\gamma} &= \lambda_{\gamma}(g_{\alpha\beta}+\theta_{\alpha\beta})dx^{\gamma}+[\mu_{\gamma}g_{\alpha\beta}+(\mu_{\gamma}+2\ell_{\gamma})\theta_{\alpha\beta}]\bar{D}\ell^{\gamma} \\
&+2\theta_{\alpha\beta}\ell_{\kappa}(\Gamma^{*\kappa}_{o\gamma}-L^{-1}N^{\kappa}_{\gamma})dx^{\gamma}+2\theta_{\alpha\beta}\ell_{\kappa}(A^{\kappa}_{o\gamma}-M^{\kappa}_{\gamma})\bar{D}\ell^{\gamma}.
\end{aligned}
\tag{1.21}
$$

From (1.21) we get

(1.22) $$g_{\alpha\beta|\gamma}=\lambda_{\gamma}(g_{\alpha\beta}+\theta_{\alpha\beta})+2\theta_{\alpha\beta}\ell_{\kappa}(\Gamma^{*\kappa}_{\sigma\gamma}-L^{-1}N^{\kappa}_{\gamma})$$

(1.23) $$g_{\alpha\beta}\Big|_{\gamma} = \mu_{\gamma}g_{\alpha\beta}+(\mu_{\gamma}+2\ell_{\gamma})\theta_{\alpha\beta} +2\theta_{\alpha\beta}\ell_{\kappa}(A^{\kappa}_{\sigma\gamma}-M^{\kappa}_{\gamma}).$$

From (1.22), (1.12) and (1.16b) we obtain (1.17). From (1.23), (1.13) and (1.16c) we get (1.18).

Definition 1.2. The D recurrent Finsler space in which the connection coefficients are given by (1.17) and (1.18) will be denoted by $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(N,M)$.

Special cases of connection coefficients are obtained if we take $(N^{\alpha}_{\gamma},0)$, $(L\Gamma^{*\alpha}_{\sigma\gamma},A^{\alpha}_{\sigma\gamma})$ or $(L\Gamma^{*\alpha}_{\sigma\gamma},0)$ instead of an arbitrarily chosen pair $(N^{\alpha}_{\gamma},M^{\alpha}_{\gamma})$ which satisfies (1.8) and (1.9).

Lemma 1.1. With respect to the coordinate transformation (1.5) and (1.6) $\Gamma^{*\beta}_{\alpha\gamma}$ and $\Gamma^{*\beta}_{\sigma\gamma}$ (both homogeneous of degree zero in $\dot{x}$) are transformed in the following way:

(1.24) $$\Gamma^{*\alpha'}_{\beta'\gamma'} = \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} +\Gamma^{*\alpha}_{\beta\gamma}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}$$

(1.25) $$\Gamma^{*\alpha'}_{\sigma\gamma'} = \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\ell^{\beta}+\Gamma^{*\alpha}_{\sigma\gamma}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}$$

and $A_{\alpha\beta\gamma}$ is a tensor.

Proof. By direct calculation using (1.8), (1.9) and the known transformation law of $\gamma^{\beta}_{\alpha\gamma}$, we obtain (1.24) and (1.25). As can be seen from (1.18), $A_{\alpha\beta\gamma}$ is the sum of tensors, so it is also a tensor.

Lemma 1.2. In $F_N(D)$, we have

(1.26) $$Dg^{\alpha\beta} = -Kg^{\alpha\beta}$$

Proof. As $g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$ we obtain

$$(1.27)\qquad (Dg^{\alpha\beta})g_{\beta\gamma}+g^{\alpha\beta}Dg_{\beta\gamma} = D\delta^{\alpha}_{\gamma}.$$

From

$$D\delta^{\alpha}_{\gamma} = d\delta^{\alpha}_{\gamma}+(\Gamma^{*\alpha}_{\kappa\beta}\delta^{\kappa}_{\gamma}-\Gamma^{*\kappa}_{\gamma\beta}\delta^{\alpha}_{\kappa})dx^{\beta}+(A^{\alpha}_{\kappa\beta}\delta^{\kappa}_{\gamma}-A^{\kappa}_{\gamma\beta}\delta^{\alpha}_{\kappa})\bar{D}\ell^{\beta} = 0$$

and (1.27) we obtain (1.26).

Theorem 1.2. In $F_N(D)$ the parallel displacement is affine, i.e. the angle between two vectors does not change.

Proof. Let us denote by $|\xi|$ the length of vector ξ^{α}. Then, using (1.14), we have

$$D|\xi|^2 = 2|\xi|D|\xi| = D(g_{\alpha\beta}\xi^{\alpha}\xi^{\beta}) = Kg_{\alpha\beta}\xi^{\alpha}\xi^{\beta}+2g_{\alpha\beta}(D\xi^{\alpha})\xi^{\beta}.$$

As by the parallel displacement $D\xi^{\alpha}=D\xi^{\beta}=0$, we obtain

$$2|\xi|D|\xi| = Kg_{\alpha\beta}\xi^{\alpha}\xi^{\beta} = K|\xi|^2 \Rightarrow$$

$$(1.28)\qquad D|\xi| = 2^{-1}K|\xi|.$$

If $\theta: \measuredangle(\xi,\eta)$, where ξ^{α} and η^{α} are two vector fields, then

$$D(g_{\alpha\beta}\xi^{\alpha}\eta^{\beta}) = D(|\xi||\eta|\cos\theta) =$$

$$(|\eta|D|\xi|+|\xi|D|\eta|)\cos\theta+|\xi||\eta|D\cos\theta.$$

Using (1.14) and (1.28) for the parallel displacement of vectors ξ^{α} and η^{α}, we obtain

$$D(g_{\alpha\beta}\xi^{\alpha}\eta^{\beta})=Kg_{\alpha\beta}\xi^{\alpha}\eta^{\beta} = K|\xi||\eta|\cos\theta = K|\xi||\eta|\cos\theta+|\xi||\eta|D\cos\theta \Rightarrow$$

$$D\cos\theta = 0.$$

§ 2. CONNECTION COEFFICIENTS IN THE SPACE $F_N(D,\lambda,\mu,\theta,\Gamma,A)(N,M)$ WHERE $L^{-1}N^\alpha_\beta = \Gamma^{*\alpha}_{0\beta}$ AND $M^\alpha_\beta = A^\alpha_{0\beta}$.

Since for any tensor field $T^\alpha_\beta(x,\dot{x})$ in $F_N(D)$ we have (1.2), it is most natural that $\bar{D}\ell^\alpha$ instead of (1.4) be defined in the following form

$$D\ell^\alpha = d\ell^\alpha+\Gamma^{*\alpha}_{0\beta}dx^\beta+A^\alpha_{0\beta}D\ell^\beta. \tag{2.1}$$

In this case $\Gamma^*_{\alpha\beta\gamma}$, $A_{\alpha\beta\gamma}$ determined by (1.17), (1.18) respectivly have another form, because here

(2.2) a) $L^{-1}N^\alpha_\beta = \Gamma^{*\alpha}_{0\beta}$ b) $M^\alpha_\beta = A^{\ \alpha}_{0\beta}$.

Lemma 2.1. Under condition (2.2a), the connection coefficients $\Gamma^*_{\alpha\beta\gamma}$ are functions of λ, θ and $\tilde{\Gamma}$, determined by (2.3)-(2.6). Under condition (2.2b), $A_{\alpha\beta\gamma}$ are functions of μ, θ and $\tilde{A}$, determined by (2.7)-(2.10).

Proof. From (1.17) and (2.2a) we have

$$\Gamma^*_{\alpha\beta\gamma} = {}^1\Gamma^*_{\alpha\beta\gamma}+{}^2\Gamma^*_{\alpha\beta\gamma}+{}^3\Gamma^*_{\alpha\beta\gamma}+{}^4\Gamma^*_{\alpha\beta\gamma}, \tag{2.3}$$

where

(2.4)

(a) $${}^1\Gamma^*_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma}-2^{-1}Q_{\alpha\beta\gamma}({}^1\Gamma^*)$$

(b) $${}^2\Gamma^*_{\alpha\beta\gamma} = -2^{-1}(\lambda_\gamma g_{\alpha\beta}+\lambda_\alpha g_{\beta\gamma}-\lambda_\beta g_{\gamma\alpha})-2^{-1}Q_{\alpha\beta\gamma}({}^2\Gamma^*)$$

(c) $${}^3\Gamma^*_{\alpha\beta\gamma} = -2^{-1}(\lambda_\gamma\theta_{\alpha\beta}+\lambda_\alpha\theta_{\beta\gamma}-\lambda_\beta\theta_{\gamma\alpha})-2^{-1}Q_{\alpha\beta\gamma}({}^3\Gamma^*)$$

(d) $${}^4\Gamma^*_{\alpha\beta\gamma} = -2^{-1}(\tilde{\Gamma}_{\alpha\beta\gamma}-\tilde{\Gamma}_{\beta\gamma\alpha}+\tilde{\Gamma}_{\gamma\alpha\beta})-2^{-1}Q_{\alpha\beta\gamma}({}^4\Gamma^*)$$

$$Q_{\alpha\beta\gamma}({}^k\Gamma^*) = L(\dot{\partial}_\delta g_{\alpha\beta}\,{}^k\Gamma^{*\delta}_{0\gamma}+\dot{\partial}_\delta g_{\beta\gamma}\,{}^k\Gamma^{*\delta}_{0\alpha}-\dot{\partial}_\delta g_{\gamma\alpha}\,{}^k\Gamma^{*\delta}_{0\beta}) \tag{2.5}$$

$k = 1,2,3,4.$

From (2.3), (2.4) and (2.5) it follows that

$$
(2.6)\qquad
\begin{aligned}
&(a)\ {}^{1}\Gamma^{*}_{o\beta\gamma} = \gamma_{o\beta\gamma} - 2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{1}\Gamma^{*\delta}_{oo}\\
&(b)\ {}^{1}\Gamma^{*}_{o\beta o} = \gamma_{o\beta o}\\
&(c)\ {}^{2}\Gamma^{*}_{o\beta\gamma} = -2^{-1}(\lambda_{\gamma}\ell_{\beta} + \lambda_{o}g_{\beta\gamma} - \lambda_{\beta}\ell_{\gamma}) - 2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{2}\Gamma^{*\delta}_{oo}\\
&(d)\ {}^{2}\Gamma^{*}_{o\beta o} = -2^{-1}(2\lambda_{o}\ell_{\beta} - \lambda_{\beta})\\
&(e)\ {}^{3}\Gamma^{*}_{o\beta\gamma} = -2^{-1}(\lambda_{\gamma}\theta_{o\beta} + \lambda_{o}\theta_{\beta\gamma} - \lambda_{\beta}\theta_{\gamma o}) - 2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{3}\Gamma^{*\delta}_{oo}\\
&(f)\ {}^{3}\Gamma^{*}_{o\beta o} = -2^{-1}(2\lambda_{o}\theta_{o\beta} - \lambda_{\beta}\theta_{oo})\\
&(g)\ {}^{4}\Gamma^{*}_{o\beta\gamma} = -2^{-1}(\tilde{\Gamma}_{o\beta\gamma} - \tilde{\Gamma}_{\beta\gamma o} + \tilde{\Gamma}_{\gamma o\beta}) - 2^{-1}L\dot{\partial}_{\delta}g_{\alpha\beta}\,{}^{4}\Gamma^{*\delta}_{oo}\\
&(h)\ {}^{4}\Gamma^{*}_{o\beta o} = \tilde{\Gamma}_{oo\beta}.
\end{aligned}
$$

From (1.18) and (2.2b) we obtain

$$
(2.7)\qquad A_{\alpha\beta\gamma} = {}^{1}A_{\alpha\beta\gamma} + {}^{2}A_{\alpha\beta\gamma} + {}^{3}A_{\alpha\beta\gamma} + {}^{4}A_{\alpha\beta\gamma},
$$

where

$$
(2.8)\qquad
\begin{aligned}
&(a)\ {}^{1}A_{\alpha\beta\gamma} = 2^{-1}L(\dot{\partial}_{\delta}g_{\alpha\beta}\delta^{\delta}_{\gamma} + \dot{\partial}_{\delta}g_{\beta\gamma}\delta^{\delta}_{\alpha} - \dot{\partial}_{\delta}g_{\gamma\alpha}\delta^{\delta}_{\beta}) - 2^{-1}T_{\alpha\beta\gamma}({}^{1}A)\\
&(b)\ {}^{2}A_{\alpha\beta\gamma} = -2^{-1}(\mu_{\gamma}g_{\alpha\beta} + \mu_{\alpha}g_{\beta\gamma} - \mu_{\beta}g_{\alpha\gamma}) - 2^{-1}T_{\alpha\beta\gamma}({}^{2}A)\\
&(c)\ {}^{3}A_{\alpha\beta\gamma} = -2^{-1}[((\mu_{\gamma} + 2\ell_{\gamma})\theta_{\alpha\beta} + (\mu_{\alpha} + 2\ell_{\alpha})\theta_{\beta\gamma} - (\mu_{\beta} + 2\ell_{\beta})\theta_{\alpha\gamma}] - 2^{-1}T_{\alpha\beta\gamma}({}^{3}A)\\
&(d)\ {}^{4}A_{\alpha\beta\gamma} = 2^{-1}(\tilde{A}_{\alpha\beta\gamma} - \tilde{A}_{\beta\gamma\alpha} + \tilde{A}_{\gamma\alpha\beta}) - 2^{-1}T_{\alpha\beta\gamma}({}^{4}A)
\end{aligned}
$$

$$
(2.9)\qquad T_{\alpha\beta\gamma}({}^{k}A) = L(\dot{\partial}_{\delta}g_{\alpha\beta}\,{}^{k}A^{\delta}_{o\gamma} + \dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{k}A^{\delta}_{o\alpha} - \dot{\partial}_{\delta}g_{\gamma\alpha}\,{}^{k}A^{\delta}_{o\beta})
$$

$k = 1,2,3,4.$

From (2.7), (2.8) and (2.9) it follows that

(a) ${}^{1}A_{0\beta\gamma} = 0 \Rightarrow$

(b) $T_{\alpha\beta\gamma}({}^{1}A) = 0$

(c) ${}^{2}A_{0\beta\gamma} = -2^{-1}(\mu_{\gamma}\ell_{\beta}+\mu_{0}g_{\beta\gamma}-\mu_{\beta}\ell_{\gamma})-2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{2}A^{\delta}_{00} \Rightarrow$

(d) ${}^{2}A_{0\beta 0} = -2^{-1}(2\mu_{0}\ell_{\beta}-\mu_{\beta})$

(2.10) (e) ${}^{3}A_{0\beta\gamma} = -2^{-1}[(\mu_{\gamma}+2\ell_{\gamma})\theta_{0\beta}+(\mu_{0}+2)\theta_{\alpha\beta}-(\mu_{\beta}+2\ell_{\beta})\theta_{0\gamma}]-$

$2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{3}A^{\delta}_{00} \Rightarrow$

(f) ${}^{3}A_{0\beta 0} = -2^{-1}[2(\mu_{0}+2)\theta_{0\beta}-(\mu_{\beta}+2\ell_{\beta})\theta_{00}]$,

(g) ${}^{4}A_{0\beta\gamma} = 2^{-1}(\tilde{A}_{0\beta\gamma}-\tilde{A}_{\beta\gamma 0}+\tilde{A}_{\gamma 0\beta})-2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{4}A^{\delta}_{00} \Rightarrow$

(h) ${}^{4}A_{0\beta 0} = \tilde{A}_{00\beta}$.

With respect to the coordinate transformation (1.5) and (1.6) ${}^{k}\Gamma^{*}_{\alpha\beta\gamma}$, $Q_{\alpha\beta\gamma}({}^{k}\Gamma)$, ${}^{k}\Gamma^{*}_{0\beta\gamma}$, ${}^{k}\Gamma^{*}_{0\beta 0}$ for $k = 2,3,4$, $A_{\alpha\beta\gamma}$, ${}^{k}A_{\alpha\beta\gamma}$, $T_{\alpha\beta\gamma}({}^{k}A)$, ${}^{k}A_{0\beta\gamma}$, ${}^{k}A_{0\beta 0}$ for $k = 1,2,3,4$ determined by (2.4b)-(2.5), (2.6c)-(2.6h) and (2.7)-(2.10) are transformed as tensors (because they are sums of tensors) and $\Gamma^{*\beta}_{\alpha\gamma}$, ${}^{1}\Gamma^{*\beta}_{\alpha\gamma}$, determined by (2.3), (2.4a) are transformed as connection coefficients (see 1.24) and (1.25)).

Definition 2.1. Recurrent Finsler space $F_N(D)$ supplied with the connection coefficients Γ^{*} and A which satisfy relations (2.3)-(2.10), will be denoted by $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$.

Lemma 2.2. There are 2^5 essentially different types of connection coefficients in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$.

Proof. If in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$ some of vector fields λ, μ, or some of tensor fields , $\tilde{A}$ or $\tilde{\Gamma}$ are zero, then in its place in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})$ we shall put zero. The 2^5 different types of connection coefficients in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$ are:

(2.11)

1. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})$
2. $F_N(D,0,\mu,\theta,\tilde{\Gamma},\tilde{A})$
3. $F_N(D,\lambda,0,\theta,\tilde{\Gamma},\tilde{A})$
4. $F_N(D,\lambda,\mu,0,\tilde{\Gamma},\tilde{A})$
5. $F_N(D,\lambda,\mu,\theta,0,\tilde{A})$
6. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},0)$
7. $F_N(D,0,0,\theta,\tilde{\Gamma},\tilde{A})$
8. $F_N(D,0,\mu,0,\tilde{\Gamma},\tilde{A})$
9. $F_N(D,0,\mu,\theta,0,\tilde{A})$
10. $F_N(D,0,\mu,\theta,\tilde{\Gamma},0)$
11. $F_N(D,\lambda,0,0,\tilde{\Gamma},\tilde{A})$
12. $F_N(D,\lambda,0,\theta,0,\tilde{A})$
13. $F_N(D,\lambda,0,\theta,\tilde{\Gamma},0)$
14. $F_N(D,\lambda,\mu,0,0,\tilde{A})$
15. $F_N(D,\lambda,\mu,0,\tilde{\Gamma},0)$
16. $F_N(D,\lambda,\mu,\theta,0,0)$
17. $F_N(D,0,0,0,\tilde{\Gamma},\tilde{A})$
18. $F_N(D,0,0,\theta,0,\tilde{A})$
19. $F_N(D,0,\mu,0,0,\tilde{A})$
20. $F_N(D,\lambda,0,0,0,\tilde{A})$
21. $F_N(D,0,0,\theta,\tilde{\Gamma},0)$
22. $F_N(D,0,\mu,0,\tilde{\Gamma},0)$
23. $F_N(D,\lambda,0,0,\tilde{\Gamma},0)$
24. $F_N(D,0,\mu,\theta,0,0)$
25. $F_N(D,\lambda,0,\theta,0,0)$
26. $F_N(D,\lambda,\mu,0,0,0)$
27. $F_N(D,\lambda,0,0,0,0)$
28. $F_N(D,0,\mu,0,0,0)$
29. $F_N(D,0,0,\theta,0,0)$
30. $F_N(D,0,0,0,\tilde{\Gamma},0)$
31. $F_N(D,0,0,0,0,\tilde{A})$
32. $F_N(D,0,0,0,0,0)$

We shall use the notations:

$$ {}^{k}\Gamma = \left[\left({}^{k}\bar{\Gamma}^{*}_{0\beta 0}=0 \right) \wedge \left({}^{k}\ddot{\Gamma}^{*}_{0\beta\gamma}=0 \right) \wedge \left({}^{k}\Gamma^{*}_{\alpha\beta\gamma}=0 \right) \right] \quad k=2,3,4 \tag{2.12} $$

$$ {}^{k}A = \left[\left({}^{k}A_{0\beta 0}=0 \right) \wedge \left({}^{k}A_{0\beta\ddot{\gamma}}=0 \right) \wedge \left({}^{k}A_{\alpha\beta\gamma}=0 \right) \right] \quad k=1,2,3,4 \tag{2.13} $$

$$ \begin{aligned} {}^{3}\bar{A} = \Big[& \left({}^{3}A_{0\beta 0}=-(2\theta_{\alpha\beta}-\ell_{\beta}\theta_{\sigma 0} \right) \wedge \\ & \left({}^{3}A_{0\beta\gamma}=-(\ell_{\gamma}\theta_{0\beta}+\theta_{\gamma\beta}-\ell_{\beta}\theta_{0\gamma})-2^{-1}\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{3}A^{\delta}_{00} \right) \wedge \\ & \left({}^{3}A_{\alpha\beta\gamma}=-(\ell_{\gamma}\theta_{\alpha\beta}+\ell_{\alpha}\theta_{\beta\gamma}-\ell_{\beta}\theta_{\alpha\gamma})-2^{-1}T_{\alpha\beta\gamma}({}^{3}A) \right) \Big] \end{aligned} \tag{2.14} $$

Lemma 2.3. In the special cases of $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_\sigma,A_\sigma)$ given by (2.11) the connection coefficients which are different from (2,3)-(2.10) are denoted in the following list (where the notations from (2.12)-(2.14) are used):

1.
2. ${}^2\Gamma, {}^3\Gamma$
3. ${}^2A, {}^3\bar{A}$
4. ${}^3\Gamma, {}^3A$
5. ${}^4\Gamma$
6. ${}^4A$
7. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3\bar{A}$
8. ${}^2\Gamma, {}^3\Gamma, {}^3A$
9. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma$
10. ${}^2\Gamma, {}^3\Gamma, {}^4A$
11. ${}^3\Gamma, {}^2A, {}^3A$
12. ${}^4\Gamma, {}^2A, {}^3\bar{A}$
13. ${}^2A, {}^3\bar{A}, {}^4A$
14. ${}^3\Gamma, {}^4\Gamma, {}^3A$
15. ${}^3\Gamma, {}^3A, {}^4A$
16. ${}^4\Gamma, {}^4A$
17. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3A$
18. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3\bar{A}$
19. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^3A$
20. ${}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A$
21. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3\bar{A}, {}^4A$
22. ${}^2\Gamma, {}^3\Gamma, {}^3A, {}^4A$
23. ${}^3\Gamma, {}^2A, {}^3A, {}^4A$
24. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^4A$
25. ${}^4\Gamma, {}^2A, {}^3\bar{A}, {}^4A$
26. ${}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A, {}^4A$
27. ${}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A, {}^4A$
28. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^3A, {}^4A$
29. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3\bar{A}, {}^4A$
30. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3A, {}^4A$
31. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A$
32. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A, {}^4A$.

Theorem 2.1. In the Finsler space there are 2^7 essentially different types of connection coefficients which by parallel displacement preserve the angle between two vectors.

Proof. As we proved in Theorem 1.2. in the space $F_N(D)$ the angle between two vectors by parallel displacement does not change. The connection coefficients in $F_N(D)$ which satisfy (1.2)-(1.4), (1.10)-(1.15), (1.20) depend on arbitrarily chosen seven parameters: λ_γ, $\mu_\gamma, \theta_{\alpha\beta}, \tilde{\Gamma}^\beta_{\alpha\gamma}, \tilde{A}^\beta_{\alpha\gamma}, N^\alpha_\beta, M^\alpha_\beta$. Instead of the arbitrary pair $(N^\alpha_\beta, M^\alpha_\beta)$, we can choose

$(L\Gamma^{*\alpha}_{\sigma\beta}, A^\alpha_{\sigma\beta})$, $(L\Gamma^{*\beta}_{\sigma\gamma}, 0)$, $(N^\alpha_\beta, 0)$, so we have

a. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(N,M)$

b. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$

c. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(N,0)$

d. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,0)$.

Since $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})$ may take 2^5 different forms given in (2.11), so we have $4\cdot 2^5=2^7$ essentially different kinds of connection coefficients which depend on the choice of (N,M), $(\Gamma_0,A_0)(N,0)$ or $(\Gamma_0,0)$.

It is clear from the above that all the mentioned connection coefficients are special cases of a.1. The first introduced and most examined case is d.32. which is $F_N(D,0,0,0,0,0)(\Gamma_0,0)$. It is the Finsler space supplied with the well known Cartan connection coefficients. We obtain them from (1.17) and (1.18), if we put them in

$$\lambda_\gamma=0,\mu_\gamma=0,\theta_{\alpha\beta}=0,\tilde{\Gamma}_{\alpha\beta\gamma}=0,\tilde{A}_{\alpha\beta\gamma}=0,N^\alpha_\beta=L\Gamma^\alpha_{0\beta},M^\alpha_\beta=0.$$

These connection coefficients appear in the papers of E.Cartan, O.Varga, in the book by H.Rund [9] and many other authors.

Parameters λ_γ and μ_γ were first introduced by A.Moór in |6|. He examined special cases of b.4. i.e. such a space

$F_N(D,\lambda,\mu,0,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$ in which $\Gamma^*_{\alpha\beta\gamma}=\Gamma^*_{\beta\alpha\gamma}$ and $A_{\alpha\beta\gamma}=A_{\beta\alpha\gamma}$.

The nonlinear connection N^α_β has recently been introduced and appears in a book by M.Matsumoto [4]. He studied in |5| the space c.17., the space $F_N(D,0,0,0,\Gamma,A)(N,0)$ in which $\lambda_\gamma=0,\mu_\gamma=0,\theta_{\alpha\beta}=0$ and $M^\alpha_\beta=0$.

Parameter θ was first introduced by the present author and some special cases of b. were studied in [1], [2].

In this paper the tensor M^α_β is introduced for the first time, and the most general till now known connection coefficients are obtained by (1.17) and (1.18).

§ 3. SOME FUNDAMENTAL RELATIONS IN F_N $(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})$ (N,M)

From

$$g_{\alpha\beta}g^{\beta\delta}=\delta_\alpha^\delta,$$

we have

$$(3.1)\qquad g_{\alpha\beta\,|\gamma}g^{\beta\delta}+g^{\beta\delta}{}_{|\gamma}g_{\alpha\beta}=0$$

$$(3.2)\qquad g_{\alpha\beta}|_{\gamma}\,g^{\beta\delta}+g^{\beta\delta}|_{\gamma}g_{\alpha\beta}=0.$$

From (3.1) and (1.22) we get

$$(3.3)\qquad g^{\beta\delta}{}_{|\gamma}=-\lambda_\gamma(g^{\beta\delta}+\theta^{\beta\delta})\quad-2\theta^{\beta\delta}(\Gamma^{*\,o}_{o\gamma}-L^{-1}N^{o}_{\gamma}).$$

From (3.2) and (1.23) we have

$$(3.4)\qquad g^{\beta\delta}|_{\gamma}=-\mu_\gamma g^{\beta\delta}-(\mu_\gamma+2\ell_\gamma)\theta^{\beta\delta}\quad-2\theta^{\beta\delta}(A^{o}_{o\gamma}-M^{o}_{\gamma}).$$

In (3.3) and in (3.4)

$$\theta^{\beta\delta}=g^{\beta\kappa}g^{\delta\rho}\theta_{\kappa\rho}.$$

From (3.3), (3.4) and (1.26) we have

$$(3.5)\qquad Dg^{\beta\delta}=g^{\beta\delta}{}_{|\gamma}dx^\gamma+g^{\beta\delta}|_{\gamma}\bar{D}\ell^\gamma=-(\lambda_\gamma dx^\gamma+\mu_\gamma\bar{D}\ell^\gamma)g^{\beta\delta}.$$

By direct calculation,we obtain

$$(3.6)\qquad \dot{x}^\delta{}_{|\gamma}=\partial_\gamma\dot{x}^\delta-(\dot{\partial}_\kappa\dot{x}^\delta)N^\kappa_\gamma+\Gamma^{*\;\delta}_{\kappa\;\gamma}\dot{x}^\kappa=\Gamma^{*\delta}_{\gamma}-N^\delta_\gamma\;,\quad(\Gamma^{*\,\delta}_{\gamma}=\Gamma^{*\delta}_{\beta\gamma}\dot{x}^\beta)$$

Introducing the notation $I^\kappa_\gamma=\delta^\kappa_\gamma-M^\kappa_\gamma$, we have

$$(3.7)\qquad \dot{x}^\delta|_\gamma=L(\dot{\partial}_\kappa\dot{x}^\delta)I^\kappa_\gamma+A_\kappa{}^\delta{}_\gamma\dot{x}^\kappa=LI^\delta_\gamma+A_\kappa{}^\delta{}_\gamma\dot{x}^\kappa.$$

If we put

$$(3.8)\qquad \ell_\delta|_\gamma=L(\dot{\partial}_\iota\ell_\delta)I^\iota_\gamma-A_\delta{}^\kappa{}_\gamma\ell_\kappa,$$

then from

$$L\dot{\partial}_\iota \ell_\delta = L\dot{\partial}_\iota \dot{\partial}_\delta L = 2^{-1}\dot{\partial}_\iota \dot{\partial}_\delta L^2 - \dot{\partial} L\dot{\partial}_\delta L$$

and (3.8) we obtain

$$(3.9) \qquad \ell_\delta|_\gamma = (g_{\iota\delta} - \ell_\iota \ell_\delta) I^\iota_\gamma - A_\delta{}^\kappa{}_\gamma \ell_\kappa .$$

For $\ell^\delta|_\gamma$ we have

$$\ell^\delta|_\gamma = (g^{\delta\kappa}\ell_\kappa)|_\gamma = g^{\delta\kappa}|_\gamma \ell_\kappa + g^{\delta\kappa}\ell_\kappa|_\gamma .$$

Substituting (3.4) and (3.9) into the above equation, we obtain

$$(3.10) \qquad \ell^\delta|_\gamma = -\mu_\gamma \ell^\delta - (\mu_\gamma + 2\ell_\gamma)\theta^{0\delta} + (\delta^\delta_\iota - \ell_\iota \ell^\delta) I^\iota_\gamma - g^{\delta\kappa} A_\kappa{}^0{}_\gamma - 2(A^0_{0\gamma} - M^0_\gamma)\theta^{0\delta} .$$

For any scalar function $G(x,\dot{x})$, we shall introduce the notations

$$(3.11) \qquad G_{|\gamma} = \partial_\gamma G - (\dot{\partial}_\iota G) N^\iota_\gamma$$

$$(3.12) \qquad G|_\gamma = L(\dot{\partial}_\kappa G) I^\kappa_\gamma .$$

We have

$$DG = dG = \partial_\gamma G dx^\gamma + \dot{\partial}_\gamma G d\dot{x}^\gamma .$$

Substituting $d\dot{x}^\iota$ from (1.4) we get

$$(3.13) \qquad dG = \partial_\gamma G dx^\gamma + L(\dot{\partial}_\iota G) I^\iota_\gamma \bar{D}\ell^\gamma - L(\dot{\partial}_\iota G)\dot{x}^\iota \, dL^{-1} - (\dot{\partial}_\iota G) N^\iota_\gamma dx^\gamma ,$$

from which it follows that

$$DG = dG = G_{|\gamma} dx^\gamma + G|_\gamma \bar{D}\ell^\gamma$$

is true only when $G(x,\dot{x})$ is homogeneous to degree zero in $\dot{x}$.
For $G(x,\dot{x}) = L(x,\dot{x})$ we have

$$(3.14) \qquad L_{|\gamma} = \partial_\gamma L - (\dot{\partial}_\iota L) N^\iota_\gamma = \partial_\gamma L - N^0_\gamma$$

$$(3.15) \qquad L|_\gamma = L(\dot{\partial}_\kappa L) I^\kappa_\gamma = L(\ell_\gamma - M^0_\gamma).$$

Since $L(x,\dot{x})$ is homogeneous to degree one in $\dot{x}$, then from (3.13) follows

$$dL = L_{|\gamma} dx^{\gamma} + L_{|\gamma} \bar{D}\ell^{\gamma} + dL.$$

From the above equation, we obtain

$$L_{|\gamma} dx^{\gamma} + L_{|\gamma} \bar{D}\ell^{\gamma} = 0. \tag{3.16}$$

From

$$L^{2} = g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta}, \tag{3.17}$$

we obtain

$$2LL_{|\gamma} = g_{\alpha\beta|\gamma} \dot{x}^{\alpha} \dot{x}^{\beta} + 2g_{\alpha\beta} \dot{x}^{\alpha}{}_{|\gamma} \dot{x}^{\beta}.$$

Substituting (1.22) and (3.6) into above equation, we obtain

$$L_{|\gamma} = 2^{-1} L\lambda_{\gamma}(1+\theta_{oo}) + \ell_{\alpha}(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})(1+\theta_{oo}). \tag{3.18}$$

In a similar way from (3.17) using (1.23) and (3.7), we get

$$L_{|\gamma} = 2^{-1} L\mu_{\gamma} + 2^{-1} L(\mu_{\gamma} + 2\ell_{\gamma})\theta_{oo} + L\ell_{\gamma} + L(A_{o}{}^{o}{}_{\gamma} - M^{o}_{\gamma})(1+\theta_{oo}). \tag{3.19}$$

From

$$\dot{x}^{\delta}{}_{|\gamma} = L_{|\gamma} \ell^{\delta} + L\ell^{\delta}{}_{|\gamma}$$

(3.6) and (3.18), we obtain

$$\ell^{\delta}{}_{|\gamma} = L^{-1}(\Gamma^{*\delta}_{\gamma} - N^{\delta}_{\gamma}) - \ell^{\delta} 2^{-1} \lambda_{\gamma}(1+\theta_{oo}) - L^{-1} \ell^{\delta} \ell_{\alpha}(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})(1+\theta_{oo}),$$

i.e.

$$\ell^{\delta}{}_{|\gamma} = L^{-1}(\delta^{\delta}_{\alpha} - \ell^{\delta}\ell_{\alpha})(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma}) - 2^{-1} \ell^{\delta} \lambda_{\gamma}(1+\theta_{oo}) - L^{-1} \ell^{\delta} \ell_{\alpha}(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})\theta_{oo}. \tag{3.20}$$

Using (1.22) and (3.20) we have

$$\ell_{\delta|\gamma} = (g_{\delta\kappa} \ell^{\kappa})_{|\gamma} = g_{\delta\kappa|\gamma} \ell^{\kappa} + g_{\delta\kappa} \ell^{\kappa}{}_{|\gamma}$$

$$\begin{aligned} \ell_{\delta|\gamma} = {} & \lambda_{\gamma}\ell_{\delta} + \lambda_{\gamma}\theta_{o\delta} - 2^{-1} \ell_{\delta}\lambda_{\gamma}(1+\theta_{oo}) + L^{-1}(g_{\delta\alpha} - \ell_{\delta}\ell_{\alpha})(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma}) \\ & + L^{-1}(-\theta_{oo}\ell_{\delta}\ell_{\alpha} + 2\theta_{o\delta}\ell_{\alpha})(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma}) \end{aligned} \tag{3.21}$$

Relations (3.14) and (3.18), further more (3.15) and (3.19) are consistent, which can be proved in the following way: From (1.17) we obtain

$$(3.22)\qquad \Gamma^{*}_{o o\gamma}=\Gamma^{*}_{\alpha\beta\gamma}\ell^{\alpha}\ell^{\beta}=\gamma_{oo\gamma}-2^{-1}\lambda_{\gamma}(1+\theta_{oo})-\theta_{oo}(\Gamma^{*o}_{o\gamma}-L^{-1}N^{o}_{\gamma}),$$

because

$$\tilde{\Gamma}_{\gamma oo}-\tilde{\Gamma}_{o\gamma o}+\tilde{\Gamma}_{\gamma oo}=0.$$

From (1.1) we have

$$\gamma_{oo\gamma}=\gamma_{\alpha\beta\gamma}\ell^{\alpha}\ell^{\beta}=$$

$$2^{-1}L^{-2}[\partial_{\gamma}\dot{\partial}_{\alpha}\dot{\partial}_{\beta}(2^{-1}L^{2})+\partial_{\alpha}\dot{\partial}_{\beta}\dot{\partial}_{\gamma}(2^{-1}L^{2})-\partial_{\beta}\dot{\partial}_{\alpha}\dot{\partial}_{\gamma}(2^{-1}L^{2})]\dot{x}^{\alpha}\dot{x}^{\beta}.$$

Using the homogeneity condition, we obtain

$$(3.23)\qquad \gamma_{oo\gamma}=L^{-1}\partial_{\gamma}L.$$

Using the relation

$$\ell_{\alpha}\Gamma^{*\alpha}_{\gamma}=L\Gamma^{*}_{oo\gamma},$$

further substituting (3.22) and (3.23) into (3.18), we obtain (3.14).

On the other side from (1.18), we have

$$(3.24)\qquad (1+\theta_{oo})(A^{o}_{o\gamma}-M^{o}_{\gamma})=-2^{-1}\mu_{\gamma}-2^{-1}(\mu_{\gamma}+2\ell_{\gamma})\theta_{oo}-M^{o}_{\gamma}.$$

Substituting (3.24) into (3.19), we obtain (3.15).

From (3.18) and (3.19) we get

$$(3.25)\qquad L_{,\gamma}dx^{\gamma}+L_{|\gamma}\bar{D}\ell^{\gamma}=2^{-1}L(1+\theta_{oo})[\lambda_{\gamma}+L^{-1}2\ell_{\alpha}(\Gamma^{*\alpha}_{\gamma}-N^{\alpha}_{\gamma})]dx^{\gamma}+$$

$$2^{-1}L(1+\theta_{oo})[\mu_{\gamma}+2\ell_{\gamma}+2(A^{o}_{o\gamma}-M^{o}_{\gamma})]\bar{D}\ell^{\gamma},$$

Using (1.20) we obtain

$$(3.26)\qquad L_{|\gamma}dx^{\gamma}+L_{|\gamma}\bar{D}\ell^{\gamma} = 0.$$

In the space d. 32. i.e. in $F_N(D,0,0,0,0,0)(\Gamma_0,0)$ where $\lambda_{\gamma} = 0$, $\mu_{\gamma}= 0$, $\theta_{\alpha\beta}= 0$, $\tilde{\Gamma}^{*}_{\alpha\beta\gamma} = 0$, $\tilde{A}_{\alpha\beta\gamma}= 0$, $N^{\alpha}_{\beta} =\Gamma^{*\alpha}_{\beta} = L\Gamma^{*\alpha}_{0\beta}$, $M^{\alpha}_{\beta} = 0$ $I^{\kappa}_{\gamma} = \delta^{\kappa}_{\gamma}$, $A_{0}{}^{\kappa}{}_{\beta}= 0$, $A^{\cdot\,0}_{\alpha\ \beta}= 0$, $A_{0}{}^{\alpha}{}_{0} = 0$, we have from (3.6), (3.7), (3.9), (3.10), (3.18), (3.19), (3.20), (3.21) respectively

$$\dot{x}^{\delta}{}_{\shortmid\gamma} = 0, \qquad \dot{x}^{\delta}{}_{|\gamma} = L\delta^{\delta}_{\gamma}$$

$$\ell_{\delta\shortmid\gamma} = g_{\delta\gamma} - \ell_{\delta}\ell_{\gamma}$$

$$\ell^{\delta}{}_{|\gamma} = \delta^{\delta}_{\gamma} - \ell^{\delta}\ell_{\gamma}$$

$$L_{\shortmid\gamma} = 0, \quad L_{|\gamma} = L\ell_{\gamma}, \quad \ell^{\delta}{}_{\shortmid\gamma}= 0, \quad \ell_{\delta|\gamma} = 0,$$

which are the well known relations in the Finsler space supplied by the Cartan connection coefficients.

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REZIME

AFINE KONEKCIJE U FINSLEROVOM PROSTORU

U Finslerovom prostoru su uvedeni generalisani koeficijenti koneksije koji zavise od sedam proizvoljno izabranih parametara. Specijalni slučajevi ovih su Cartanovi koeficijenti koneksije, kao i oni iz rekurentnih Finslerovih prostora. Svi imaju osobinu da se pri paralelnom pomeranju ugao izmedju vektora ne menja. Pokazano je da postoji 2^7 bitno različitih tipova koneksija ove vrste. Za generalni slučaj su nadjene neke fundamentalne relacije.

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A UNIFORMLY CONVERGENT DISCRETIZATION METHOD FOR A SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM OF THE FOURTH ORDER

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ABSTRACT

In this paper we consider problem (1) with a small parameter $\varepsilon>0$ and the basic assumption that $a(x)>0$. A numerical method of Petrov-Galerkin type is proposed and exponential splines as test-functions are used. Using the approach from [4] the linear convergence, uniform in ε, of the method is proved.

1. INTRODUCTION

We consider the singulary perturbed boundary value problem

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(1) $\varepsilon u^{(4)}(x)-(a(x)u'(x))'=f(x), u(0)=u'(0)=u(1)=u'(1)=0$

with some small parameter $0<\varepsilon<<1$. Let us suppose $a(x)\geqslant\alpha>0$, the functions $a(x), f(x)$ are assumed to be smooth. Whereas in the literature a large number of results on second order problems exists (compare [1],[5] and the references cited therein) only few facts are known on fourth order equations. Having used a maximum principle argument, Shishkin [6] obtained some uniform with respect to the parameter ε convergence results for a problem which can be splitted immediately into a system of two second order equations. The difficulties connected with fourth order problems are due to the fact that in general a maximum principle does not hold. Our approach is based on ideas of O'Riordan and Stynes [4] and does not use any maximum principle neither for the solvability of the discrete problem nor in the convergence proof.

It is well-known that in the case of second order equations

(2) $-\varepsilon u''(x)+c(x)u'(x)+d(x)u(x)=f(x)$

the uniform convergence of the Il'in-scheme or the El-Mistikawy-Werle scheme at first was proved under the assumption $d(x)\equiv 0$. Analogously we hope that the more general problem

$$\varepsilon u^{(4)}-(ax)u')'+b(x)u=f(x)$$

can be handled without complete exponential fitting discretizing the trem $b(x)u$ completely analogously to $f(x)$. Our discretization method is a Petrov-Galerkin method using exponential splines as test functions.

Let us denote by $\|\cdot\|_k$ the norm in the Sobolev space $H^k(0,1)$ and by $\|\cdot\|_\infty$ the maximum norm. Let $(.,.)$ denote the scalar product in $L^2(0,1)$. set

$$U=\{v \in H^2(0,1) \text{ with } v(0)=v'(0)=v(1)=v'(1)\}.$$

A weak formulation of (1) is: Find $u \in U$ such that

(3) $\quad a(u,v):=\varepsilon(u'',v'')+(au',v')=(f,v)$, for all $v \in U$.

The bilinear form $a(.,.)$ is continuous and coercive - therefore (3) admits a unique solution.

Setting $v=u$ in (3) we obtain

(4) $\quad \| u\|_1 \leq \frac{1}{\alpha} \sqrt{3/2}\| f\|_0$.

Thus our problem (3) is in some sense ε-uniformly stable, the imbedding $H^1 \to C$ yields

(5) $\quad \| u\|_\infty \leq K\| f\|_\infty$,

too. Here and in the following we denote by K in general different constants which do not depend on ε.
According to (4) a usual conform finite element method results in an ε-uniformly stable scheme (in the sense of (4) or (5)). But one con not expect uniform convergence because the solution of our original problem contains boundary layer functions. It holds [3]

(6) $\quad u(x,\varepsilon)=g(x,\varepsilon)+\varepsilon^{\frac{1}{2}}g_0(x,\varepsilon)\exp(-\sqrt{a(0)}x/\varepsilon^{\frac{1}{2}})+$

$\quad\quad +\varepsilon^{\frac{1}{2}}g_1(x,\varepsilon)\exp(-\sqrt{a(1)}(1-x)/\varepsilon^{\frac{1}{2}})$

where the functions g can be expanded in an asymptotic power series with respect to $\varepsilon^{1/2}$, further it is possible to differentiate (6). In particular, $z(x)=g(x,0)$ solves the reduced problem

$$-(a(x)z')'=f(x),\ z(0)=z(1)=0.$$

From (6) we obtain the bounds

$$(7)\quad \text{(i)}\ \|u\|_\infty \leq K, \text{(ii)} \|u^{(1+\ell)}\|_\infty \leq K(\varepsilon^{-1/2})^\ell \quad (\ell=0,1,2,\ldots)$$

$$\text{(iii)} \|u\|_k \leq K\ \varepsilon^{-1/4}(\varepsilon^{-1/2})^{k-1} \quad (k=2,3,\ldots).$$

Thus, the standard error analysis for conform finite element methods using piecewise quadratic or piecewise cubic C^1-splines would result in

$$\|u-u_h\|_2 \leq \begin{cases} Kh\varepsilon^{-7/4} & \text{for quadratic } C^1\text{-splines} \\ Kh^2\varepsilon^{-9/4} & \text{for cubic } C^1\text{-splines} \end{cases}$$

Of course, for $\varepsilon \ll 1$ this estimate is practically not applicable and it is necessary to use special discretization techniques.

2. The discrete problem

Let some nonequidistant grid be given, i.e.

$$0=x_o<x_1<\ldots<x_{N-1}<x_N=1.$$

The corresponding step sizes we donote by $h_i=x_i-x_{i-1}$, the mesh width by $h = \max\limits_{1\leq i\leq N} h_i$. We define by $\bar{a}(x)$ a piecewise constant approximation of a(x) with $\bar{a}(x)=a_i:=a(x_i)$ for all $x\in(x_{i-1},x_i)$ and modify our original bilinear from due to

$$(8)\quad \bar{a}(u,v):=\varepsilon(u'',v'')+(\bar{a}u',v').$$

Let some finite-dimensional spaces of the same dimension S_h,T_h with $S_h\subset U$, $T_h\subset U$ be given. The discrete problem consists in the following: Find $u_h \in S_h$ such that

(9) $\bar{a}(u_h,v_h)=(\bar{f},v_h)$ for all $v_h \in T_h$,

where $\bar{f}$ denotes a corresponding piecewise constant approximation to f. Now we introduce basic functions $\psi_k (k=1,\ldots,N-1)$ in $T_h \subset U$ by

(10) $\varepsilon \psi_k^{(4)} - \bar{a}\psi_k'' = 0$ on (x_{i-1},x_i) for $i=1,2,\ldots,N$

$\psi_k \in C^2[0,1]$ with supp $\psi_k=[x_{k-2},x_{k+2}]$

$\psi_k(x_k)=1$.

Thus, the test functions ψ_k are exponential C^2-splines and it is well known that ψ_k are uniquely determined (later we have to calculate them explicitely). Let us for a moment choose $S_h=T_h$ and set

$$u_h = \sum_{k=1}^{N-1} v_k \psi_k .$$

Then the unique solvability of the continuous problem results in the unique solvability of the discrete problem, therefore the corresponding system of linear equations for the unknowns v_k admits a unique solution. First we are intersted in computing approximations in the gridpoints. In principle one could use

(11) $u_h(x_k)=u_k=\psi_k(x_{k-1})v_{k-1}+v_k+\psi_k(x_{k+1})v_{k+1}$

to calculate these approximations. But it is easier to derive a linear system of equations for u_k itselves.

According to the definition of our discrete problem the function $u_h \in S_h \subset U$ (S_h arbitrary) satisfies

$$(12) \quad (-\varepsilon\psi_k'' + \bar{a}\psi_k', u_h') = (f, \psi_k).$$

Taking into account the properties of our test functions it holds

$$(13) \quad -\varepsilon\psi_k + \bar{a}\psi_k' = c_i^k \text{ on } [x_{i-1}, x_i]$$

with $c_i = 0$ for $i \leq k-2$, $i \geq k+3$.

Hence, (12) results in

$$(14) \quad \sum_{i=k-1}^{i=k+2} c_i^k (u_i - u_{i-1}) = d_k^* = (\bar{f}, \psi_k) \quad (k=1,\ldots,N-1)$$

(with $c_{N+1}^{N-1} = 0$). Whereas the system for v_k represents a seven band matrix our system (14) for u_k is pentadiagonal. The system (14) does not depend on the special choice of the space S_h, thus for $S_h = T_h$ we also obtain (14) and so this system admits a unique solution, too.

Now let us set

$$t = (x - x_{i-1})/h_i, \quad \lambda_i^2 = a_i/\varepsilon, \quad \mu_i = h_i \lambda_i.$$

Then it is convinient to represent $\psi_k(x)$ on the interval $[x_{i-1}, x_i]$ in the form [2]

$$(15) \quad s_i^k(x) = y_{i-1}^k (1-t) + y_i^k t + \frac{M_{i-1}^k}{\lambda_i^2} \left(\frac{\sinh \mu_i (1-t)}{\sinh \mu_i} - (1-t) \right) + \frac{M_i^k}{\lambda^2} \left(\frac{\sinh \mu_i t}{\sinh \mu_i} - t \right),$$

and ψ_k belongs to C^2 iff

$$(16)\quad \rho_i M^k_{i-1}+(\sigma_i+\sigma_{i+1})M^k_i+\rho_{i+1}M^k_{i+1}=\tau^k_{i+1}-\tau^k_i \quad (i=1,\dots,N-1)$$

with $\tau^k_i=(y^k_i-y^k_{i-1})/h_i$

$$\rho_i = \frac{\sinh\mu_i-\mu_i}{\mu_i^2\sinh\mu_i}\,h_i, \quad \sigma_i = \frac{\mu_i\cosh\mu_i-\sinh\mu_i}{\mu_i^2\sinh\mu_i}\ .$$

The differentiation results in

$$(17)\qquad c^k_i = a_i\,\frac{y^k_i-y^k_{i-1}}{h_i}+\varepsilon\,\frac{M^k_{i-1}-M^k_i}{h_i}\ .$$

Unfortunately, it is not so easy to calculate y^k_i, M^k_i explicitely. In the case $2\leq k\leq N-2$ it holds $M^k_\ell=0$ for $\ell\leq k-2$ and $\ell\geq k+2$, $y^k_\ell=0$ for $\ell\leq k-2$, $\ell\geq k+2$ and $y_k=1$, the five parameters M^k_{k-1}, M^k_k, M^k_{k+1}, y^k_{k-1}, y^k_{k+1} satisfy (16) for i = k-2, k-1, k, k+1, k+2. A tedious but simple computation yields: y^k_{k-1}, y^k_{k+1} are the solutions of

$$(18)\quad \frac{1}{\rho_{k+1}}\Big[\frac{1}{h_{k+1}}+\frac{1}{h_{k+2}}+(\sigma_{k+1}+\sigma_{k+2})\frac{1}{\rho_{k+2}h_{k+2}}\Big]y^k_{k+1}-\frac{1}{\rho_k}\Big[\frac{1}{h_k}+\frac{1}{h_{k-1}}+$$

$$+(\sigma_{k-1}+\sigma_k)\frac{1}{\rho_{k-1}h_{k-1}}\Big]y^k_{k-1} = \frac{1}{\rho_{k+1}h_{k+1}}-\frac{1}{\rho_k h_k}$$

$$\overset{*}{p}_k y^k_{k+1}+\overset{*}{q}_k y^k_{k-1}=2\Big(\frac{1}{h_k}+\frac{1}{h_{k+1}}\Big)+(\sigma_k+\sigma_{k+1})\Big(\frac{1}{\rho_k h_k}+\frac{1}{\rho_{k+1}h_{k+1}}\Big)$$

with

$$p_k^* = \frac{2}{h_{k+1}} - 2\,\frac{\rho_k}{\rho_{k+2}}\,\frac{1}{h_{k+2}} + \frac{\sigma_k+\sigma_{k+1}}{\rho_{k+1}}\left[\frac{1}{h_{k+2}} + \frac{1}{h_{k+2}} + (\sigma_{k+1}+\sigma_{k+2})\frac{1}{\rho_{k+2}h_{k+2}}\right]$$

$$q_k^* = \frac{2}{h_k} - 2\,\frac{k}{\rho_{k-1}}\,\frac{1}{h_{k-1}} + \frac{\sigma_k+\sigma_{k+1}}{\rho_k}\left[\frac{1}{h_k} + \frac{1}{h_{k-1}} + (\sigma_{k-1}+\sigma_k)\,\frac{1}{\rho_{k-1}h_{k-1}}\right].$$

If y_{k-1}^k, y_{k+1}^k are known we obtain M_ℓ^k by the formulas

$$(19)\ \ (i)\ \ M_{k-1}^k = \frac{1}{\rho_{k-1}h_{k-1}}\,y_{k-1}^k,\quad M_{k+1}^k = \frac{1}{\rho_{k+2}h_{k+2}}\,y_{k+1}^k$$

$$(ii)\ \ M_k^k = \frac{1}{\rho_k h_k} - \frac{1}{\rho_k}\left[\frac{1}{h_k} + \frac{1}{h_{k-1}} + (\sigma_{k-1}+\sigma_k)\,\frac{1}{\rho_{k-1}h_{k-1}}\right]y_{k-1}^k.$$

For an equidistant grid and $a(x)=a=\text{const.}$ the ρ_i and σ_i are constant and we get

$$(20)\ \ (i)\ \ y_{k-1}^k = y_{k+1}^k = \rho/(2\sigma)$$

$$(ii)\ \ M_{k-1}^k = M_{k+1}^k = 1/(2h\sigma),\quad M_k^k = -1/(h\sigma).$$

It is easy to see that for the order in h it holds

$$y_{k-1}^k,\ y_{k+1}^k = O(1),\quad M_\ell^k = O(h^{-2})\quad (\ell = k-1, k, k+1)$$

corresponding to the fact that the term $a_i(y_i^k - y_{i-1}^k)/h_i$ in (17) generates the discretization of $-(au')'$ and the trem $(M_{i-1}^k - M_i^k)/h_i$ generates the discretization of $\varepsilon u^{(4)}$ (the right-hand side of our discrete problem (14) admits the order $O(h)$).

The basic functions $\psi_1(x)$ and $\psi_{N-1}(x)$ contain the unknown parameters $M_o^1, M_1^1, M_2^1, y_2^1$ (respectively M_{N-2}^{N-1}, M_{N-1}^{N-1}, M_N^{N-1}, y_{N-2}^{N-1}) and we have to take into consideration the boundary conditions. We obtain

$$(21)\quad \rho_1 M_o^1 - \{\frac{1}{h_2} - \frac{\rho_2}{\rho_3}\frac{1}{h_3} + \frac{\sigma_1+\sigma_2}{\rho_2}[\frac{1}{h_2} + \frac{1}{h_3} + \frac{\sigma_2+\sigma_3}{\rho_3 h_3}]\} y_2^1 = -(\frac{1}{h_1} + \frac{1}{h_2}) -$$

$$-\frac{\sigma_1+\sigma_2}{\rho_2}\frac{1}{h_2}\sigma_1 M_o^1 - \frac{\rho_1}{\rho_2}[\frac{1}{h_2} + \frac{1}{h_3} + \frac{\sigma_2+\sigma_3}{\rho_3 h_3}] y_2^1 = \frac{1}{h_1} - \frac{\rho_1}{\rho_2}\frac{1}{h_2}\,.$$

If M_o^1, y_2^1 are known we get

$$(22)\quad M_2^1 = \frac{1}{\rho_3 h_3} y_2^1 \;;\; M_1^1 = \frac{1}{\rho_2 h_2} - \frac{1}{\rho_2}[\frac{1}{h_2} + \frac{1}{h_3} + \frac{\sigma_2+\sigma_3}{\rho_3 h_3}]\, y_2^1.$$

Substituting $M_k^1 := M_{N-k}^{N-1}$ $(k=0,1,2)$, $y_2^1 := y_{N-2}^{N-1}$, $h_{k+1} := h_{N-k}$, $\rho_{k+1} := \rho_{N-k}$, $\rho_{k+1} := \sigma_{N-k}$ the corresponding system for the basic function $\psi_{N-1}(x)$ is generated. For an equidistant grid with a(x) a it holds

$$(23)\quad (i)\;\; y_2^1 = \rho\sigma/(2\sigma^2-\rho^2)$$

$$(ii)\;\; M_o^1 = +2(\sigma+\rho)/(h(-\rho^2+2\sigma^2)),\; M_1^1 = -(2\sigma+\rho)/(h(2\sigma^2-\rho^2)),$$

$$M_1^2 = \sigma/(h(2\sigma^2-\rho^2))$$

Now, let us set

$$(24)\qquad d_k^* = \sum_{i=k-1}^{i=k+2} d_i^k f_i$$

for the right-hand side of our discrete problem (14). We have

$$d_i^k = \int_{x_{i-1}}^{x_i} s_i^k(x)dx = h_i \int_0^1 s_i(t)dt$$

and on such a way we get

$$(25)\quad d_i^k = h_i \frac{y_i^k + y_{i-1}^k}{2} + \frac{M_i^k + M_{i-1}^k}{\lambda_i^2}\left(\frac{\cosh\mu_i - 1}{\mu_i \sinh\mu_i} - \frac{1}{2}\right).$$

Thus, all coefficients of our discrete problem (14) can be computed using (17),(24),(25) and the formulas (18),(19), (21), (22) for the y_ℓ^k, M_ℓ^k ($\ell = k-1, k, k+1$).

3. The uniform convergence of the method

Let us define the selfadjoint operator L on the function space U by

$$(Lu,v) = (u,Lv) := \bar{a}(u,v).$$

The solution of the problem

$$L\,G_j = \delta(x-x_j),$$

a Green function, is characterized by

$$(26)\quad \varepsilon(G_j'', w'') + (\bar{a}G_j', w') = w(x_j) \text{ for all } x \in U.$$

It holds $G_j \in H^3$ and $G_j \in C^2$. Choosing for w some finite funcions we obtain

$$(27)\quad \varepsilon G_j^{(4)} - \bar{a}G_j'' = 0 \text{ on every subinterval } (x_{i-1}, x_i).$$

Combining (26) and (27) we get that G_j satisfies the following jump condition

$$(28)\quad S_i G_j := \lim_{x\to x_i-0}(\varepsilon G_j''' - \overline{a}G_j') - \lim_{x\to x_i+0}(\varepsilon G_j''' - \delta\overline{a}G_j') = -\delta_{ij}.$$

Thus, G_j is equivalently characterized by $G_j \in C^2[0,1]$,

$G_j(0)=G_j'(0)=G_j(1)G_j'(1)=0$ and (27),(28).

Lemma 1: G_j *belongs to the test space* T_h.

Proof: All functions in T_h can be represented as

$$H_h = \sum_{k=1}^{N-1} \alpha_k \psi_k .$$

New, we have to prove only that it is possible to fulfil the jump condition (28) choosing the parameters α_k in an adequate way. Let us choose $\varphi \in T_h$. From (10) it follows

$$\sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (\varepsilon\psi_k^{(4)} - \overline{a}\psi_k'')\varphi \, dx = 0.$$

Integration by parts results in

$$\sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} (\sigma\psi_k''' - \overline{a}_k')\varphi \Big|_{x_{i-1}}^{x_i} - \sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (\varepsilon\psi_k''' - \overline{a}\psi_k)\varphi' \, dx = 0,$$

respectively

$$(29)\quad \sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} (S_i\psi_k)\varphi(x_i) + \sum_{k=1}^{N} \alpha_k [(\varepsilon\psi_k'', \varphi'') + (\overline{a}\psi_k', \varphi')] = 0.$$

Let $\varphi_m (m=1,..,N-1)$ be some basic function of T_h with

$$\varphi_m(x_\ell) = \delta_{\ell m}.$$

Choosing $\varphi = \varphi_m$ in (29) we get

$$(30)\quad S_m H_j + \sum_{k=1}^{N-1} \alpha_k[(\varepsilon\psi_k'',\varphi_m'')+(\bar{a}\psi_k',\varphi_m')]=0.$$

Now, one can choose α_k in such a way that $S_m H_j=-\delta_{mj}$ **because** (30) represents a linear system for the unknowns α_k which is equivalent to the system for the unknowns v_k and therefore uniquely solvable. □

Lemma 2: *For the error at the grid points it holds*

$$(31)\quad u(x_i)-u_i=(u',(\bar{a}-a)G_i')+(f-\bar{f},G_i),$$

Proof: Let us start with

$$u(x_i)-u_i((u-u_h)(x),\delta(x-x_i))=((u-u_h)(x),LG_i)=\bar{a}(u-u_h,G_i).$$

A splitting yields

$$\begin{aligned}u(x_i)-u_i&=\bar{a}(u,G_i)-\bar{a}(u_h,G_i)\\&=\bar{a}(u,G_i)-a(u,G_i)+a(u,G_i)-\bar{a}(u_h,G_i)\\&=((\bar{a}-a)u',G_i')+(f,G_i)-(\bar{f},G_i)\end{aligned}$$

taking into consideration $G_i\in T_h$. □

THEOREM 1: *Let us assume*

$$\|\bar{a}-a\|_\infty<Kh,\ \|\bar{f}-f\|_\infty<Kh.$$

Then, the above defined method converges uniformly with respect to ε *at the gridpoints and the following error estimation holds*

$$(32)\quad \max_i|u(x_i)-u_i|<Kh,$$

where K *does not depend on* ε, h.

Proof: Let us note that we have already proved $\|u\|_1 \leq K$. Analogously, from (26) it follows $\|G_i\|_1 \leq K, \|G_i\|_\infty \leq K$. Now, the statement follows directly from (31).□

In order to approximate u(x) for all x we use the linear interpolate πu_h which satisfies $(\pi u_h)(x_i)=u_i$.

Lemma 3: *For the linear interpolate it holds*

$$\|u-\pi u_h\|_\infty \leq Kh$$

that means, u is approximated uniformly with respect to ε in the L^∞ *-norm.*

Proof: Let us fix some $x \in (x_{i-1}, x_i)$. Then we have

$$u(x)-\pi u_h(x)=[u(x)-u(x_{i-1})+u(x_{i-1})-u_{i-1}](x-x_{i-1})/(x_i-x_{i-1})+$$

$$+[u(x)-u(x_i)+u(x_i)-u_i](x-x_i)/(x_{i-1}-x_i).$$

Thus, from theorem 1 and $\|u'\|_\infty \leq K$ the desired result follows.□

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REZIME

JEDAN UNIFORMNO KONVERGENTAN METOD DISKRETIZACIJE ZA SINGULARNO PERTURBOVANI KONTURNI PROBLEM ČETVRTOG REDA

U radu se posmatra problem (1) sa malim parametrom $\varepsilon>0$ i osnovnom pretpostavkom $a(x)>0$. Predložen je numerički postupak Petrova-Galjerkina koji koristi eksponencijalne splajnove kao test-funkcije. Korišćenjem pristupa iz rada [4] za postupak je dokazana linearna konvergencija, uniformna po ε.

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S-ASYMPTOTIC OF SOLUTIONS OF THE ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT

The S-asymptotic behaviour of solutions of non-homogeneous elliptic partial differential equations is given on a ray $\{\rho w, \rho>0\}$, where $w \in R^n$ and $\|w\|=1$.

INTRODUCTION.

The S-asymptotic is one of the notions related to the asymptotical behaviour of distributions elaborated in [3] . It can be profitable in many studies and applications (see [5]), especially in the quantum field theory. In [1] one can find references to papers from the quantum field theory which pushed forward the study of the S-asymptotic behaviour of distributions.

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Key words: S-asymptotic of a distribution,fundamental solution.

1. FUNDAMENTAL SOLUTIONS OF AN ELLIPTIC DIFFERENTIAL EQUATION

A fundamental solution of a differential equation

$$P(i\,\frac{\partial}{\partial x})u=0$$

in n-dimensions is a distribution $G\in D'(R^n)$ satisfying the equation

$$P(i\,\frac{\partial}{\partial x})G(x)=\delta(x).$$

Let $P_o(i\,\frac{\partial}{\partial x})$ be the principal part of $P(i\,\frac{\partial}{\partial x})$. If $P_o(y)\neq 0$ for any real $y\neq 0$, then $P(i\,\frac{\partial}{\partial x})$ is called an elliptic operator. Assume that the coefficients of P are real numbers. If the elliptic operator is also homogeneous (i.e. if it coincides with its principal part), then fundamental solution G(x) has the form:

$$(1)\quad G(x)=\begin{cases} A(\frac{x}{r})\,r^{m-n}, & \text{for } n \text{ odd, or } n \text{ even and } m<n \\ B(\frac{x}{r})\,r^{m-n}+C(x)\,\ln r, & \text{for } n \text{ even and } m\geqslant n\ 1. \end{cases}$$

where A(y) and B(y) are analytic functions on $\|y\|=1$ and C(x) is a polynomial in x of degree m-n ; m is the degree of the principal polynomial $P_o(y)$ (m is then necessarily an even number) [2]; $r^2=\|x\|^2=\sum_{i=1}^{n} x_i^2$.

2. S-ASYMPTOTIC OF FUNDAMENTAL SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

We shall use the following definition of the S-asymptotic i S'[3] :

Definition. Distribution $T\in S'$ has the S-asymptotic related to $c(h)>0$ in a cone Γ with the limit $U\in S'$ if and only if there exists the limit:

$$(2)\qquad \lim_{h\in\Gamma,\|h\|\to\infty} \langle T(x+h)/c(h),\varphi(x)\rangle=\langle U,\varphi\rangle$$

for every $\varphi \in S$. If $\Gamma \equiv \{\rho w, \rho>0\}, \|w\|=1$, we write in short:

$$T(x+\rho w) \overset{S}{\sim} c(\rho)U \ , \rho\to\infty.$$

Our aim it to find S-asymptotic of the fundamental solution G given by relation (1).

First we shall treat the function $A(\frac{x}{r})\, r^{m-n}$, $m>n$; $A(y)$ is analytic on $\|y\|=1$. Γ will be the ray $\{\rho w, \rho>0\}$ for a fixed $w \in R^n$, $\|w\|=1$. In this case limit (2) is

$$J \equiv \lim_{\rho\to\infty} <\rho^{n-m} A((x+\rho w)/\| x+\rho w \|)\| x+\rho w\|^{m-n}\| , \varphi(x)>$$

$$= \lim_{\rho\to\infty} \int_{R^n} A((\frac{x}{\rho}+w)/\|\frac{x}{\rho}+w\|)\|\frac{x}{\rho}+w\|^{m-n}\varphi(x)\ dx \ .$$

Since

$$A((\frac{x}{\rho}+w)/\|\frac{x}{\rho}+w\|)\|\frac{x}{\rho}+w\|^{m-n} <$$

$$< M\ [\sum_{i=1}^{n}(|x_i|+|w_i|)^2\ ^{(m-n)/2}]\ , \rho>1$$

where $M=\sup A(y)$, $\|y\|=1$, we can use the Labegue´s Lemma which gives

$$J=A(w/\|w\|)\|w\|^{m-n}\int_{R^n}\varphi(x)\ dx = A(w)\int_{R^n}\varphi(x)\ dx.$$

We proved that $A(\frac{x}{r})\, r^{m-n}$, $m>n$, has the S-asymptotic related to ρ^{m-n} on the ray $\{\rho w, \rho>0\}$, $\|w\|=1$ and with the limit $U=A(w)$.

The next step is to find the S-asymptotic of the function $A(\frac{x}{r})\, r^{m-n}$, but in case $m<n$, $m>0$.

Suppose that in relation (2) the function φ belongs to D and supp $\varphi \subset I^n$, $I=(-a,a), a\in R_+$. Then for $\rho \geq a/\max|w_i|$, $i=1,\dots,n$

$$\lim_{\rho\to\infty} \rho^{n-m}\int_{R^n} A((x+\rho w)/\|x+\rho w\|)\|x+\rho w\|^{m-n}\varphi(x)\ dx$$

$$=A(w)\int_{R^n}\varphi(x)\ dx.$$

Hence, $A(\frac{x}{r})r^{m-n}$, $m<n$, has the S-asymptotic in D´ with the limit A(w). Since D is dense in S, to prove that this func-

tion has the same S-asymptotic in S', it suffices to prove that $\rho^{n-m}J(\rho)$, $\rho>0$, is bounded (see the Banach-Steinchaus theorem), where

$$J(\rho)=\int_{R^n} A\left(\frac{x+\rho w}{\|x+\rho w\|}\right)\|x+\rho w\|^{m-n}\varphi(x)\,dx\ ,\ \varphi\in S.$$

Therefore, we shall divide the integral $J(\rho)$ into three parts and use the following inequalities:

For $\|x\|\leq\rho/2$ and $\|x\|\geq 3\rho/2$, $\|x+\rho w\|\geq|\,\|x\|-\rho\,|\geq\rho/2$.
For $\varphi\in S$ and $\rho/2\leq\|x\|\leq 3\rho/2$, $|\varphi(x)|\leq K(1+\|x\|^2)^{-k}\leq K(1+(\rho/2)^2)^{-k}$, where k is any integer and K is a constant.

Now, we have for $\varphi\in S$, $m<n$, $m>0$:

$$\rho^{n-m}\left|\int_{\|x\|<\frac{\rho}{2}}+\int_{\|x\|>\frac{3\rho}{2}} A\left(\frac{x+\rho w}{\|x+\rho w\|}\right)\|x+\rho w\|^{m-n}\varphi(x)\,dx\right|<$$

$$<2\;2^{n-m}M\int_{R^n}|\varphi(x)|\,dx,$$

where $M=\sup|A(w)|$, $\|w\|=1$, and

$$\left|\int_{\|x\|>\frac{\rho}{2}}^{\|x\|<\frac{3\rho}{2}} A\left(\frac{x+\rho w}{\|x+\rho w\|}\right)\|x+\rho w\|^{m-n}\varphi(x)\,dx\right|<$$

$$<K\frac{M}{(1+\frac{\rho^2}{4})^k}\int_{\|x\|>\frac{\rho}{2}}^{\|x\|<\frac{3\rho}{2}}\|x+\rho w\|^{m-n}\,dx\to 0,\ \rho\to\infty.$$

Assume, now, that $C(x) = \sum\limits_{|i|\leq m-n} \alpha_i x^i$ $m \geq n$, where $\alpha_i \in R$; $i=(i_1,\dots,i_n)$; $x^i = x_1^{i_1}\dots x_n^{i_n}$; $|i|=i_1+\dots i_n$. Then,

$$\lim_{\rho\to\infty} \rho^{n-m}(\ln\rho)^{-1}<C(x+\rho w)\ \ln\|x+w\|,\varphi(x)>=$$

$$= \lim_{\rho\to\infty} \rho^{n-m}(\ln\rho)^{-1} \int_{R^n} \sum_{|i|\leq m-n} \alpha_i\ (x+\rho w)^i(\ln\rho +$$

$$+ \ln\|\frac{x}{\rho} + w\|)\varphi(x)\ dx = \sum_{|i|=m-n} \alpha_i\ w^i \int_{R^n}\varphi(x)\ dx$$

by the same reason as in the first case.

The results of this paragraph can be expressed in the following

Proposition 1. If G is a fundamental solution of the eliptic homogeneous operator P of degree m in n-dimension, then on the ray $\{\rho w,\rho>0\},\|w\|=1$, G has an S-asymptotic:

$$G(x+\rho w) \overset{s}{\sim} \rho^{m-n}\cdot G(w),\ \rho\to\infty \text{ for n odd, or n even and } m<n\ ;$$

(3) $$G(x+\rho w) \overset{s}{\sim} \rho^{m-n}\ln\rho\cdot D(w),\ \rho\to\infty, \text{ for n even and } m\geq n,$$

where $D(w)= \lim\limits_{\rho\to\infty} G(\rho w)/(\rho^{m-n}\ln\rho)$.

2. S-ASYMPTOTIC OF SOLUTIONS OF A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

Now, we can give the S-asymptotic behaviour of solutions of the partial differential equation:

(4) $$P(\ i\frac{\partial}{\partial x})u(x)=f(x),\ \ f\in O_c'$$

where P is an elliptic homogeneous operator of degree m in n-dimensions. We know that a solution of equation (4) is of the form $u=G*f$ (G being the fundamental solution of the operator P) and belongs to S'. Namely, since G belongs to S' and f to O'_c, then $G*f$ exists and belongs to S' [4]. Moreover the mapping: (G,f) $G*f$ is separately continuous.

Proposition 2. Equation (4) has a solution $u=G*f$ belonging to S'. u has an S-asymptotic on the ray $\{\rho w, \rho>0\}, \|w\|=1, \rho\to\infty$:

$$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\cdot(G(w)*f), \text{ for n odd, or n even and } m<n$$

$$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\ln\rho\cdot(D(w)*f), \text{ for n even and } m\geqslant n, \tag{5}$$

where G is the fundamental solution of the elliptic differential equation $P(i\frac{\partial}{\partial x})G(x)=\delta(x)$ and $D(w)= \lim\limits_{\rho\to\infty} G(\partial w)/(\rho^{m-n}\ln\rho)$. P is a homogeneous operator of degree m in n-dimensions

Proof. Suppose that G is a fundamental solution of the equation $P(i\frac{\partial}{\partial x})u(x)=\delta(x)$, then G has the form given in (1) and has the S-asymptotic given by relation (3). A solution of equation (4) is $u=G*f$. Since this convolution is separately continuous, from (3) follows (5).

REMARK. It is well known that if v is a solution of equation (4) and u-v has the convolution with G, then v=u. This fact says about the class of distributions in which our solution $u=G*f$ is unique.

If P is elliptic but not at any rate homogeneous, then the fundamental solution G(x) of (4) has the from

$$A(\tfrac{x}{r},r)r^{m-n}\text{ , for n odd}$$

$$B(\tfrac{x}{r},r)r^{m-n}+C(\tfrac{x}{r},r)r^{m-n}\ln r\text{, for n even,}$$

where $A(w,r)$, $B(w,r)$, $C(w,r)$ are analytic functions in (w,r) in a neighbourhood of $\|w\|=1$, $r=0$, and $C(\frac{x}{r},r)r^{m-n}$ is a function $C(x)$ which is analytic in x at $x=0$ (see [2]).

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REZIME

S-ASIMPTOTIKA REŠENJA PARCIJALNE DIFERENCIJALNE JEDNAČINE ELIPTIČKOG TIPA

U radu je dokazano sledeće tvrdjenje:

Jednačina (4) ima rešenje $u=G*f$ koja pripada skupu S'. To rešenje u ima nad zrakom $\{\rho w, \rho>0\}, \|w\|=1$ S-asimptotiku kada $\rho\to\infty$:

$$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\cdot(G(w)*f), \quad \text{za } n \text{ neparno, ili } n \text{ parno i } m<n$$

$$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\ln\rho\cdot(D(w)*f), \quad \text{za } n \text{ parno i } m\geqslant n$$

gde je G fundamentalno rešenje jednačine (4), a $D(w)= = \lim_{\rho\to\infty} G(\rho w)/(\rho^{m-n}\ln\rho)$.

Jedinstvenost rešenja u je u klasi distribucija v koje imaju osobinu da $u-v$ ima konvoluciju sa G.

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DE HAAN'S CLASS OF DISTRIBUTIONS I

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ABSTRACT

A class of distributions, named $\pi(G)$, is defined as a generalization of de Haan's Π_g class of functions [2]. The properties and the characterisation of the class $\pi(G)$ are given. The applications of the class $\pi(G)$ are left for the second part which will be treated in the next paper.

INTRODUCTION

In a quite recently published book [1], the authors named the study of the existence of the limit

$$(1)\qquad \lim_{x\to\infty} \frac{f(\lambda x)-f(x)}{g(x)} = k(\lambda), \qquad \lambda>0,$$

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with an f measurable and g positive function, as "de Haan´s theory".

For a regular varying function g the class Π_g is the class of measurable functions satisfying (1) with

$$k(\lambda) \equiv k_\rho(\lambda) = c\begin{cases} \log\lambda, & \rho=0 \\ (\lambda^\rho-1)/\rho, & \rho\neq 0 \end{cases},$$

for some constant $c\neq 0$.

In his book [2] de Haan treated this class Π_g, but he elaborated in more detail a subclass of Π_g, namely, the class Π_g but with g as a slowly varying function and c=1. The class Π_g is a proper subclass of the famous Karamata class of regularly varying functions (see [3] and [5]) and has many applications (see [1] and [2]).

Our aim is to enlarge the class Π_g to distributions (generalized functions of L.Schwartz´s type [4]) and to point at some applications of such an enlarged class.

1. THE CLASS Π_g OF DISTRIBUTIONS

Writing $F(y)=f(e^y)$, $G(y)=g(e^y)$ and $K(y)=k(e^y)$, $y\in R$, we obtain the "additive-argument version" of (1):

$$(2) \qquad \lim_{h\to\infty} \frac{F(y+h)-F(h)}{G(h)} = K(y),\ y\in R.$$

This version suits our aim better and has been frequently used in proving the properties of elements of the class Π_g .

DEFINITION 1. Let G be a positive and measurable function. A distribution $T\in\mathcal{D}'(\mathbb{R})$ belongs to the class $\pi(G)$ if and only if for every $\varphi\in\mathcal{D}(\mathbb{R})$ and every $y\in\mathbb{R}$

$$(3)\quad \lim_{h\to\infty} <\frac{T(x+y+h)-T(x+h)}{G(h)}, \varphi(x)> = <S(x,y), \varphi(x)>$$

where $S(.,y)$ is a family of distributions which is not constant in the parameter y. If in our definition $S(.,y)$ can be constant, then we have the class $\pi_o(G)$.

First, we notice that $<T(x+y),\varphi(x)>=(T*\check{\varphi})(y)\in C^\infty(\mathbb{R})$, where $\check{\varphi}(x)=\varphi(-x)$ and the asterisk $*$ is the sign of the convolution. Now, (3) can be written in the from:

$$(4)\quad \lim_{h\to\infty} \frac{(T*\check{\varphi})(y+h)-(T*\check{\varphi})(h)}{G(h)} = (S(.,y)*\check{\varphi})(0).$$

If we denote by $F(y)=(T*\check{\varphi})(y)$ and by $K(y)=(S(.,y)*$ $*\check{\varphi})(0)$, relation (4) has just the form of (2). In such a way, we related the class $\pi(G)$ of distributions with the class Π_g of functions and through relation (4) we can use all the results which concern the class Π_g, in proving properties of the class $\pi(G)$.

PROPOSITION 1. If $T\in\pi(G)$, then $S(.,y)$ has the form $S(x,y)=k_\rho(e^y)e^{\rho x}$, $\rho\in\mathbb{R}$.

Proof. For every $z\in\mathbb{R}$ we have:

$$\lim_{h\to\infty} \frac{G(z+h)}{G(h)} <\frac{T(x+y+(z+h))-T(x+(z+h))}{G(z+h)}, \varphi(x)>$$

$$= \lim_{h\to\infty} <\frac{T(x+z+y+h)-T(x+z+h)}{G(h)}, \varphi(x)>.$$

It follows from Theorem 1.9., p.16 in [2] that there exists a $\rho\in R$ such that:

$$(6)\qquad \lim_{h\to\infty} \frac{G(z+h)}{G(h)} = e^{\rho z}, \quad z\in R.$$

Relation (5) can be written now in the form:

$$(7)\qquad e^{\rho z}<S(x,y),\varphi(x)>= S(x+z,y),\varphi(x)>.$$

Takin care of the existence of the derivative of a distribution, from relation (7) we can derive the following one: $\rho S(x,y)=D_x S(x,y)$, where $D_x S$ is the derivative in x of the distribution $S(x,y)$ for every $y\in R$. The unique solution of this equation is $S(x,y)=e^{\rho x}C(y)$, where $C(y)$ is a family of constant distributions. From Theorem 1.9., p.16 in [2] and relation (4), it follows that $<S(x,y),\varphi(x)>=k_\rho(e^y)$. Hence, $C(y)=c_1k_\rho(e^y)$ and $S(x,y)$ has the form $k_\rho(e^y)e^{\rho x}$.

REMARKS. 1) We can always suppose that G is a continuous function. Namely, if $T\in\pi(G)$, then for a $\varphi_0\in\mathcal{D}$

$$\lim_{h\to\infty}< \frac{T(x+y+h)-T(x+h)}{G(h)},\varphi_0(x)>=k_\rho(e^y)<e^{\rho x},\varphi_0(x)>.$$

We can choose a y_0 such that $k_\rho(e^{y_0})<e^{\rho x},\varphi_0(x)>= 1$. Since $G(h)>0$, $h\in R$, there exists a h_0 such that $<[T(x+y_0+h)-T(x+h)]\,\varphi_0(x)>\equiv G_1(h)>0$, $h>h_0$. The function $G_1(h)$ can be enlarged for $h<h_0: G_1(h)=G_1(h_0), h<h_0$. The function G_1 is continuous, positive and

$$\lim_{h\to\infty}< \frac{T(x+y+h)-T(x+h)}{G_1(h)},\varphi(x)>=$$

$$=\lim_{h\to\infty} < \frac{T(x+y+h)-T(x+h)}{G(h)}, \varphi(x) > \lim_{h\to\infty} \frac{G_1(h)}{G(h)}.$$

Since $\lim_{h\to\infty} G_1(h)/G(h)=1$, we can always replace G by G_1 in (3).

2) We have seen that G from Definition 1 satisfies relation (6). Write $G(h)=e^{\rho h}L(h)$, then there exists the following limit: $\lim_{h\to\infty} L(x+h)/L(h)=1, x\in \mathcal{R}$, and $L(\ln y)$ is a slowly verying function [5]. Then, we can always suppose that G has the form $G(h)=e^{\rho h}L(h)$.

PROPOSTITION 2. The limit given by relation (3) is uniform in y on every compact set belonging to $\mathcal{R}$.

Proof. This proposition follows directly from Theorem 3. 1.16., p.139 in [1], if we use form (4) of limit (3).

PROPOSITION 3. If f is a function belonging to the class Π_g, then $f(e^y)=F(y)$ defines a regular distribution $\tilde{F}$, which belongs to the class $\pi(G)$.

Proof. If f belongs to the class Π_g, f is measurable and $F(y)=f(e^y)$ is measurable, as well. As a consequence of Theorem 3.1.16.,p. 139 in [1], it follows that f is locally bounded ; then, the same property has F. F is locally integrable and defines a regular distribution $\tilde{F}$. Now, for a $\varphi\in\mathcal{D}$, supp $\varphi\subset[-r,r]$ and $y\in\mathcal{R}$

$$\lim_{h\to\infty} \left\langle \frac{\tilde{F}(x+y+h)-\tilde{F}(x+h)}{G(h)}, \varphi(x)\right\rangle =$$

$$= \lim_{h\to\infty} \int_{-r}^{r} \frac{F(x+y+h)-F(x+h)}{G(x+h)} \frac{G(x+h)}{G(h)} \varphi(x)\, dx.$$

We know (see [5]) that limit (6) is uniform in z on every compact set belonging to R. Taking care of Theorem 3.1.16 in [1], once again, we can apply Lebesgue´s theorem to the last integral and we shall obtain for the sought limit

$$= \int_{-r}^{r} k_{\rho}(e^{y}) e^{\rho x} \varphi(x)\, dx$$

which proves our proposition.

The next example shows that we can find a locally integrable function $h: R_{+}\to R$ which does not belong to any class Π_g, but the regular distribution $\tilde{h}(e^{x})$ belongs to a class $\pi(G)$. Such an example is the following:

$$h(x)=x \int_{e^{\alpha}}^{x} g(\ln u) \frac{du}{u} + x\, g(\ln x),\ x>0,$$

g is continuous, $g(x)\leq e^{n}, x\in J_n=(n-e^{-2n}, n+e^{-2n})$; $g(x)=0$, $x\notin J_n\, n\in N$. Hence, $g\in L^{1}(-\infty,\infty)$. Now,

$$\frac{h(xt)-h(t)}{t} = x\int_{e^{\alpha}}^{xt} g(\ln u) \frac{du}{u} - \int_{e^{\alpha}}^{t} g(\ln u) \frac{du}{u} +$$

$$+ x\, g(\ln x + \ln t) - g(\ln t)\ ,\ x>0$$

the two integrals have a limit when $t\to\infty$, but $x\, g(\ln x + \ln t) - g(\ln t)$ oscillates between zero and infinity. On the contrary, the function

$$h(e^x)=F(x)=e^x \int_\alpha^x g(v)\,dv + e^x g(x)= \frac{d}{dx}\left(e^x \int_\alpha^x g(v)\,dv\right)$$

defines a regular distribution belonging to $\pi(e^h)$. To show that, assume that $\varphi\in\mathcal{D}$ and supp $\varphi\subset[-r,r]$. Then, for the distribution $\tilde{F}$, limit (3) is:

$$-\lim_{h\to\infty}\int_{-r}^{r}\left[e^{x+y}\int_\alpha^{x+y+h} g(v)\,dv - e^x\int_\alpha^{x+h} g(v)\,dv\right]\varphi'(x)\,dx$$

$$=\int_\alpha^\infty g(v)\,dv\,(e^y-1)\int_{-r}^{r} e^x\varphi(x)\,dx\;,\; y\in R.$$

Proposition 3 makes precise in what sense the class of distributions $\pi(G)$ enlarges the class of functions Π_g. A distribution which is not regular, but belongs to a $\pi(G)$ is $T(x)=x_+^\lambda$, $\neq -1,-2,\dots,\lambda<-1$ (see [4]). In this case, for $\varphi\in\mathcal{D}$, supp $\varphi\subset[-r,r]$, $h>r+|y|$, limit (3) is:

$$\lim_{h\to\infty}\int_{-r}^{r}\frac{(x+y+h)^\lambda-(x+h)^\lambda}{h^{\lambda-1}}\varphi(x)\,dx=\lambda\, y\int_{-r}^{r}\varphi(x)\,dx.$$

Hence, the limit distribution $S(x,y)=\lambda y$.

PROPOSITION 4. A distribution $T\in\mathcal{D}'$ belongs to the class $\pi(G)$ if and only if for every interval $I_r=[-r,r]$ there exist numerical functions F_i, $i=1,2,\dots,m$, continuous on $[-r,\infty)$, such that for every i, $i=1,2,\dots,m$, the limit:

$$\lim_{h\to\infty}\frac{F_i(y+h)-F_i(h)}{G(h)}=k_{i,\rho}(e^y),\; k_{i,\rho}(e^y)=c_i\,\frac{e^{\rho y}-1}{\rho}$$

is uniform for $x\in I_r$, when $h\to\infty$. The restriction of the distribution T on $[-r,\infty)$ can be given in the form

$$T = \sum_{i=1}^{m} D^{j_i} F_i \,. \tag{8}$$

By D we denote the derivative in the sense of distributions.

Proof. First, we shall prove that the conditions are sufficient. Suppose that T is given by the sum (8), then limit (3) is:

$$\lim_{h\to\infty} \sum_{i=1}^{m} (-1)^{j_i} \int_{-r}^{r} \frac{F_i(x+y+h)-F_i(x+h)}{G(h)} \varphi^{(j_i)}(x)\,dx =$$

$$= \sum_{i=1}^{m} (-1)^{j_i} \int_{-r}^{r} k_{i,\rho}(e^y) e^{\rho x} \varphi^{(j_i)}(x)\,dx$$

$$= \int_{-r}^{r} k_\rho(e^y)\; e^{\rho x}\varphi(x)\,dx.$$

It follows that $T\in\pi(G)$.

Suppose, now, that $T\in\pi(G)$. Then, the set of distributions $H\equiv\{T(x+y+h)-T(x+h)/G(h), h\in[r,\infty), y\in[-r_1,r_1]\}$ is bounded in $\mathcal{D}'$. From a part of the proof of Theorem XXII, p. 51 in [4] or from a lemma proved in [6], p. 130 follows

LEMMA 1. If $t\in\pi(G)$, then for an interval $I_r\equiv[-r,r]$ and a Ω, which is a relatively compact open neighbourhood of zero in $\mathcal{R}$, there exists a $m\geqslant 0$, such that for every $\varphi,\psi\in\mathcal{D}^m_\Omega$ the function $(T*\varphi*\psi)(x)$ is continuous for $x\in[-r,\infty)$ and

$$\lim_{h\to\infty} \left[\frac{T(t+y+h)-T(t+h)}{G(h)} *\varphi(t)*\psi(t)\right](x) =$$

$$= k_\rho(e^y)\left[e^{\rho t}*\varphi(t)*\psi(t)\right](x) \;\;;$$

this limit is uniform in $x \in I_r$, $y \in I_{r'}$, $r' \in R$.

To bring to an end the proof of Proposition 4, let us take relation (VI, 6; 23) from [4]

$$T=\Delta^{2k}*(\gamma E*\gamma E*T)-2\Delta^{k}*(\gamma E*\xi*T)+(\xi*\xi*T) \tag{9}$$

where E is a solution of the iterated Laplace equation $\Delta^k E=\delta$; $\nu,\xi \in \mathcal{D}_\Omega$. We have only to choose the number k large enough so that γE belongs to $\mathcal{D}^m_\Omega$. If we denote by $F_1=\gamma E*\gamma E*T$, $F_2=\gamma E*\xi*T$ and by $F_3=\xi*\xi*T$, all the functions F_i, $i=1,2,3$, are of the from $F_i=T*\varphi_i*\psi_i$, where $\varphi_i,\psi_i \in \mathcal{D}^m_\Omega$, $i=1,2,3$.

By properties of the convolution we have:

$$\frac{F_i(y+h)-F_i(h)}{G(h)} = [(\frac{T(x+y+h)-T(x+h)}{G(h)})*\varphi_i(x)*\psi_i(x)](0)$$

and by Lemma 1 it follows that

$$\lim_{h\to\infty} \frac{F_i(y+h)-F_i(h)}{G(h)} = k_\rho(e^y)[e^x*\varphi_i(x)*\psi_i(x)](0)$$

$$= k_{i,\rho}(e^y)\ .$$

uniformly in $y \in I_{r'}$.

Proposition 4 characterizes the class $\pi(G)$ as the class of distributions T given by the sum of functions F_i, $F_i(\ln y) \in \Pi_g$, where $g(t)=G(\ln t)$.

PROPOSITION 5. If $T \in \pi(G)$ and $U \in \mathcal{E}'$, then $T*U \in \pi_o(G)$ if, in addition, $(U*e^{\rho x})(0) \neq 0$, then $T*U \in \pi(G)$, as well.

Proof. If $T \in \pi(G)$, then $(T(x+y+h)-T(x+h))/G(h)$ converges in $\mathcal{D}'$ for every $y \in R$. The convolution $T*U$, for a fixed U, is a continuous mapping $:\mathcal{D}' \to \mathcal{D}'$. By the properties of the convolution, we have:

$$\frac{(T*U)(t+y+h)-(T*U)(t+h)}{G(h)} = [\{\frac{T(x+y+h)-T(x+h)}{G(h)}\} * U(x)](t)$$

and this converges for every $y \in R$ to;

$$k_\rho(e^y)(e^{\rho x} * U(x))(t) = (e^{\rho x} * U(x))(0)e^{\rho t}k_\rho(e^y) .$$

COROLLARY. If $T \in \pi(G)$, $\rho \neq 0$, then for every $k \in N$, $D^k T \in \pi(G)$, as well. It follows from the facts that $D^k T = \delta^{(k)} * T$ and $\delta^{(k)} * e^{\rho x} \neq 0$. This property of the class $\pi(G)$ is very important for applications to differential equations and to other convolution equations.

2. SOME COMMENTS

1. One can ask the question why we started from the "additive-argument version" in defining the class $\pi(G)$? To answer this question, we have to remark that the membership to the class Π_g is a local property in the sense: If $f_1(t) = f_2(t)$, $t > t_0$ and if $f_1 \in \Pi_g$,then $f_2 \in \Pi_g$,as well. We have looked to find such a generalization of Π_g to keep that very natural property. The next proposition makes precise that the class $\pi(G)$ satisfies such a demand.

PROPOSITION 6. If for two distributions T_1 and T_2 we have $T_1 = T_2$ over an open interval (α, ∞) and T_1 belongs to $\pi(G)$, then T_2 belongs to $\pi(G)$, as well.

Proof. For a $\varphi \in \mathcal{D}$ we have $\langle T_1(x+y+h), \varphi(x) \rangle = \langle T_1(x), \varphi(x-y-h) \rangle$. Denote by $\psi(x) = \varphi(x-y-h) \in \mathcal{D}$. If the supp $\varphi \subset [-r,r]$, then the supp $\psi \subset [-r+y+h, r+y+h]$. We can find h_o such that $-r-|y|+h > \alpha$, $h \geq h_o$. For such a h_o $\langle T_1(x+z+h), \varphi(x) \rangle = \langle T_2(x+z+h), \varphi(x) \rangle$ when $|z| \leq |y|$, $h \geq h_o$. Now, it is easy to prove that $T_2 \in \pi(G)$.

The opposite of the asertion of Proposition 6 is not true. That follows from

PROPOSITION 7. Suppose that G is of the form $G(x) = L(x)$ (see remarks after Proposition 1) and that

$$\lim_{h\to\infty} \frac{T_1(x+h) - T_2(x+h)}{G(h)} = U \text{ in } \mathcal{D}',$$

where U is a constant distribution. If $T_1 \in \pi(G)$, then $T_2 \in \pi(G)$, as well.

Proof. For x,y belonging to compact sets in R we have the following relation:

$$\frac{T_2(x+y+h) - T_2(x+h)}{G(h)} = \frac{T_2(x+y+h) - T_1(x+y+h)}{G(h)} \frac{G(y+h)}{G(h)} +$$

$$+ \frac{T_1(x+h) - T_2(x+h)}{G(h)} + \frac{T_1(x+y+h) - T_1(x+h)}{G(h)}.$$

The assertion of Proposition 7 follows directly from this relation.

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REZIME

DE HAAN-OVA KLASA DISTRIBUCIJA

De Haan [2] je definisao i izučavao klasu Π_g funkcija koje predstavljaju striktnu potklasu Karamatinih regularno promenljivih funkcija [3]. Posebno se bavio klasom Π_g kada je g sporo promenljiva funkcija. Ta klasa ima veliku primenu u raznim oblastima matematike i u raznim oblastima njene primene (vidi[1] i [2]).

U ovom radu definisana je klasa distribucija $\pi(G)$ na sledeći način:

DEFINICIJA 1. Neka je G pozitivna i merljiva funkcija Distribucija $T\in\mathcal{D}'$ pripada klasi $\pi(G)$ tada i samo tada ako za svako $\varphi\in\mathcal{D}$ i svako $y\in\mathcal{R}$ postoji granica definisana relacijom (3), gde je $S(\cdot,y)$ familija konstantnih distribucija, $S(.,y)$ različita od konstante dok y prolazi skupom $\mathcal{R}$.

Dokazane su razne osobine klase $\pi(G)$. Tako je pokazano da je $S(x,y)=c(e^{\rho y}-1)e^{\rho x}, \rho\in\mathcal{R}$; da svaka funkcija $f\in\Pi_g$ definiše regularnu distribuciju $\tilde{F}(y)=f(e^y)$ koja pripada klasi $\pi(G), G(y)=g(e^y)$. Lokalno integrabilna funkcija f ne mora pripadati klasi Π_g ako $\tilde{F}(y)=f(e^y)$ pripada klasi $\pi(G)$. Okarakterisana je klasa $\pi(G)$ kao skup distribucija $T=\sum_{i\leq m} D^i F_i$ gde $F_i(\ln y)\in\Pi_g, g(h)=G(\ln h)$. Konvolucija sa elementom $U\in E'$ preslikava $\pi(G)$ u $\pi(G)$ ako $(U*e^{\rho x})(0)\neq 0$. To je važna osobina za primenu jer $\delta^{(k)}\in E'$, pa izvodi preslikavaju $\pi(G)$ u $\pi(G)$ ako je $\rho\neq 0$. Najzad je pokazano da pripadanje klasi $\pi(G)$ je lokalna osobina.

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A REMARK CONCERNING SLOWLY VARYING FUNCTIONS IN KARAMATA'S SENSE

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ABSTRACT

In this paper the author give a complete proof of a statement from the paper [1] .

1. INTRODUCTION

In [1] we were concerned with the properties of slowly varying functions in Karamata's sense i.e. of real functions L defined and positive for $x>0$, such that

$$\lim_{x\to+\infty} \frac{L(\lambda x)}{L(x)} = 1 \quad \text{for any } \lambda\in(0,+\infty).$$

In paper [3] J. Karamata defined this class of functions, and established also, using a particular example (p. 46-47), that a slowly varying function need not have the asymptotic behaviour of a monotonic function as $x\to+\infty$. In [1] we made this fact more precise by assertion 4^0 of Theorem II, by which: a slowly varying function may,

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as x→+∞, oscillate, with a finite or infinite interval of oscillation ([1], p. 133). However, proving this statement, we only constructed ([1], p. 134-135) a definite slowly varying function which oscillates between finite (and positive) limits as x→+∞, and noted finally that "one can, by suitable modifications of this example, construct a slowly varying function with an infinite interval of oscillation".

Here we shall give a complete proof of the second part of the cited statement, which has not been made in [1], and we shall also give a more precise form of this statement. Namely, we shall replace it by the following:

Proposition. For any two elements a and b of the interval $[0,+\infty]$ such that $a<b$, there exists a slowly varying function L continuous on $[0,+\infty)$ and such that

$$\underline{\lim}_{x\to+\infty} L(x)=a, \qquad \overline{\lim}_{x\to+\infty} L(x)=b.$$

Proof. In all cases considered, the slowly varying function L(x) which we shall construct for $x\geqslant 1$ will be given by an equality of the form

$$L(x)=e^{\int_1^x \frac{\varepsilon(t)}{t}\,dt},$$

where

$$\varepsilon(x)=\frac{\eta(x)}{1+\ln x} \quad (x\geqslant 1) \tag{1}$$

and the function $\eta(x)$ is continuous and bounded for $x\geqslant 1$; for $x\in[0,1]$ it is sufficient to define L(x) as a function continuous and positive on that interval. According to the known result about the representation of a slowly varying function (see, for example, [1], p.124, 0.2), the function L(x) defined in this way is certainly slowly varying and continuous on $[0,+\infty)$.

We shall first consider the case

$$0<a<b<+\infty .$$

Let us put

$$(2)\qquad \rho=\ln b-\ln a\ (>0),\quad \omega=e^{e^{\rho}-1}-1\ (>0),\quad \delta=\frac{\omega}{3}\ (>0)$$

and

$$(3)\qquad x_1=1+2\delta;\quad x_{n+1}=\delta+e^{e^{\rho}-1}\cdot(\delta+x_n)^{e^{\rho}}\ (n\in N).$$

Then,

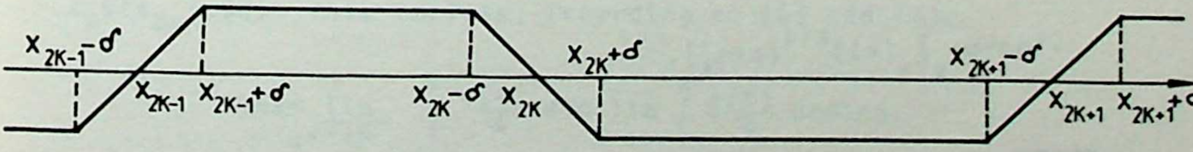
$$\frac{x_{n+1}}{x_n}>e^{e^{\rho}-1}\cdot\frac{(\delta+x_n)^{e^{\rho}}}{x_n}>e^{e^{\rho}-1};\quad \frac{\delta+x_n}{x_n}>e^{e^{\rho}-1}=\omega+1\ (>1)\ (n\in N)$$

and hence,

$$(4)\qquad x_{n+1}-x_n=x_n\left(\frac{x_{n+1}}{x_n}-1\right)>x_1\omega>\omega=3\delta\ (n\in N).$$

Let us further pose (see the figure)

$$(5)\qquad \eta(x)=\begin{cases} x-x_{2k-1} & (x_{2k-1}-\delta\leqslant x<x_{2k-1}+\delta)\\ 1 & (x_{2k-1}+\delta\leqslant x<x_{2k}-\delta)\\ -x+x_{2k} & (x_{2k}-\delta\leqslant x<x_{2k}+\delta)\\ -1 & (x_{2k}+\delta\leqslant x\leqslant x_{2k+1}-\delta)\end{cases}\quad (k\in N).$$



According to (2), (3) and (4), this definition of $\eta(x)$ is consistent and obviously $\eta(x)$ is defined by it for $x\geqslant x_1-\delta$ (>1) as a function continuous and bounded on $[0,+\infty)$. For $1\leqslant x\leqslant x_1-\delta$, we define $\eta(x)$ as a function continuous on $[1,x_1-\delta]$ and such that

$$\int_1^{x_1-\delta}\frac{\eta(t)}{t(1+\ln t)}\,dt=\ln a-\alpha-\int_{x_1-\delta}^{x_1}\frac{t-x}{t(1+\ln t)}\,dt,$$

where the number α will be determined later; evidently, this can be done.

By (3), we have

$$\int_{x_k+\delta}^{x_{k+1}-\delta}\frac{dt}{t(1+\ln t)}=\ln(1+\ln t)\Big|_{x_k+\delta}^{x_{k+1}-\delta}=\ln\frac{1+\ln(x_{k+1}-\delta)}{1+\ln(x_k+\delta)}$$

$$=\ln\frac{1+\ln[e^{e^{\rho}-1}(\delta+x_k)^{e^{\rho}}]}{1+\ln(x_k+\delta)}=\ln\frac{1+e^{\rho}-1+e^{\rho}\ln(\delta+x_k)}{1+\ln(x_k+\delta)}=\rho .$$

Hence,

$$\int_1^{x_{n+1}}\frac{\varepsilon(t)}{t}\,dt=\int_1^{x_1-\delta}\frac{\eta(t)}{t(1+\ln t)}\,dt+\int_{x_1-\delta}^{x_1}\frac{t-x_1}{t(1+\ln t)}\,dt+\sum_{k=1}^{n}\int_{x_k}^{x_{k+1}}\frac{\varepsilon(t)}{t}\,dt$$

$$=\ln a-\alpha+\sum_{k=1}^{n}(-1)^{k-1}\int_{x_k}^{x_{k+1}}\frac{|\eta(t)|}{t(1+\ln t)}\,dt$$

$$=\ln a-\alpha+\sum_{k=1}^{n}(-1)^{k-1}\Big[\int_{x_k+\delta}^{x_{k+1}-\delta}\frac{dt}{t(1+\ln t)}+\Big(\int_{x_k}^{x_k+\delta}+\int_{x_{k+1}-\delta}^{x_{k+1}}\Big)\frac{|\eta(t)|}{t(1+\ln t)}\,dt\Big]$$

$$=\ln a-\alpha+\sum_{k=1}^{n}(-1)^{k-1}(\rho+\alpha_k),$$

where

$$\alpha_n\ \overset{\text{def}}{=\!=}\ \Big(\int_{x_n}^{x_n+\delta}+\int_{x_{n+1}-\delta}^{x_{n+1}}\Big)\frac{|\eta(t)|}{t(1+\ln t)}\,dt\downarrow 0\quad(n\to\infty),$$

and consequently

$$\alpha \overset{\text{def}}{=\!=} \sum_{k=1}^{\infty} (-1)^{k-1}\alpha_k$$

is a finite number. Hence,

$$(6) \quad \int_1^{x_{2n+1}} \frac{\varepsilon(t)}{t}\, dt = \ln a - \alpha + \sum_{k=1}^{2n} (-1)^{k-1}\alpha_k \to \ln a - \alpha + \alpha = \ln a \, (n\to\infty),$$

$$(7) \quad \int_1^{x_{2n+2}} \frac{\varepsilon(t)}{t}\, dt = \ln a - \alpha + \rho + \sum_{k=1}^{2n+1} (-1)^{k-1}\alpha_k \to \ln a - \alpha + \rho + \alpha = \ln b \, (n\to\infty).$$

By (1) and (5), we have, for each $n\in N$,

$$\int_1^{x_{2n-1}} \frac{\varepsilon(t)}{t} dt < \int_1^{x} \frac{\varepsilon(t)}{t}\, dt \leq \int_1^{x_{2n}} \frac{\varepsilon(t)}{t}\, dt \quad (x_{2n-1} < x \leq x_{2n}),$$

$$\int_1^{x_{2n+1}} \frac{\varepsilon(t)}{t}\, dt \leq \int_1^{x} \frac{\varepsilon(t)}{t}\, dt < \int_1^{x_{2n}} \frac{\varepsilon(t)}{t}\, dt \quad (x_{2n} < x \leq x_{2n+1}),$$

wherefrom we obtain

$$\int_1^{y_x} \frac{\varepsilon(t)}{t}\, dt \leq \int_1^{x} \frac{\varepsilon(t)}{t}\, dt \leq \int_1^{z_x} \frac{\varepsilon(t)}{t}\, dt \quad (x > x_1),$$

where $x\in(x_k, x_{k+1}]$, $\{y_x, z_x\} = \{x_k, x_{k+1}\}$, $y_x \in \{x_{2\nu-1} : \nu\in N\}$, $z_x \in \{x_{2\nu} : \nu \in N\}$. This implies, according to (6) and (7),

$$\ln a \leq \varliminf_{x\to+\infty} \int_1^{x} \frac{\varepsilon(t)}{t}\, dt \leq \varlimsup_{x\to+\infty} \int_1^{x} \frac{\varepsilon(t)}{t}\, dt \leq \ln b,$$

and this together with (6) and (7) implies

$$\underline{\lim}_{x\to+\infty} \int_1^x \frac{\varepsilon(t)}{t}\,dt = \ln a, \qquad \overline{\lim}_{x\to+\infty} \int_1^x \frac{\varepsilon(t)}{t}\,dt = \ln b.$$

Therefore,

$$\underline{\lim}_{x\to+\infty} L(x)=a, \quad \overline{\lim}_{x\to+\infty} L(x)=b.$$

In the case

$$0<a<b=+\infty,$$

let us pose

$$x_1=3;\, x_{2n}=e^{e^n-1}(x_{2n-1}+1)^{e^n}+1\,(n\in N),\, x_{2n+1}=e^{e^n-1}(x_{2n}+1)^{e^n}+1\,(n\in N).$$

Then, we have

$$x_{2n}>1+e^{e-1}x_{2n-1}>1+2x_{2n-1}(>x_{2n-1})\ (n\in N),$$

$$x_{2n+1}>1+2x_{xn}(>x_{2n})\ (n\in N),$$

that is, $x_{n+1}>1+2x_n(>x_n)\ (n\in N)$, and so $x_{n+1}-x_n>1+x_n\geqslant1+3>3$. Therefore, in this case we shall also define $\eta(x)$ for $x\geqslant2$ by (5) with $\delta=1$, and for $x\in[1,2]$ as a continuous function such that

$$\int_1^{x_1} \frac{\eta(t)}{t(1+\ln t)}\,dt=\int_1^{x_1} \frac{\varepsilon(t)}{t}\,dt = \ln a-\alpha,$$

where

$$\alpha \stackrel{\text{def}}{=\!=} \sum_{k=1}^{\infty}(-1)^{k-1}\alpha_k\ ,$$

with

$$\alpha_k=\Big(\int_{x_k}^{x_k+1}+\int_{x_{k+1}-1}^{x_{k+1}}\Big)\frac{|\eta(t)|}{t(1+\ln t)}\,dt \quad (k\in N),$$

is a finite number. Similarly as in the first case, one can establish that now

$$\int_1^{x_{2n+1}}\frac{\varepsilon(t)}{t}\,dt=\ln a-\alpha+\sum_{k=1}^{2n}(-1)^{k-1}\alpha_k\to\ln a-\alpha+\alpha=\ln a \quad (n\to\infty),$$

$$\int_1^{x_{2n+2}}\frac{\varepsilon(t)}{t}\,dt=\ln a-\alpha+n+1+\sum_{k=1}^{2n+1}(-1)^{k-1}\alpha_k\to+\infty\,(n\to\infty),$$

wherefrom one concludes that

$$\underline{\lim}_{x\to+\infty}\int_1^x\frac{\varepsilon(t)}{t}\,dt=\ln a,\quad \overline{\lim}_{x\to+\infty}\int_1^x\frac{\varepsilon(t)}{t}\,dt=+\infty,$$

i.e. $\underline{\lim}_{x\to+\infty}L(x)=a,\quad \overline{\lim}_{x\to+\infty}L(x)=+\infty.$

In the case

$a=0<b<+\infty$,

it is sufficient to observe that the function $L(x)=1/L_0(x)$, where L_0 is a slowly varying function continuous on $[0,+\infty)$ and such that $\underline{\lim}_{x\to+\infty}L_0(x)=1/b$, $\overline{\lim}_{x\to+\infty}L_0(x)=+\infty$ (preceding case) - is continuous on $[0,+\infty)$ and has the properties $\underline{\lim}_{x\to+\infty}L(x)=0$, $\overline{\lim}_{x\to+\infty}L(x)=b$.

Let us finally consider the case

$a=0,\ b=+\infty$.

Putting

$$x_1=3;\ x_{n+1}=1+e^{e^n-1}(x_n+1)^{e^n}\qquad (n\in N),$$

and defining $\eta(x)$ on $[2,+\infty)$ by (5) with $\delta=1$, and on $[1,2]$ as a continuous function on that interval, we get in this case

$$\int_1^{x_{n+1}}\frac{\varepsilon(t)}{t}\,dt=\int_1^{x_n}\frac{\varepsilon(t)}{t}\,dt+\sum_{x=1}^{n}(-1)^{k-1}k+\sum_{k=1}^{n}(-1)^{k-1}\alpha_k,$$

where α_k has the same meaning as in the proceding case, and consequently

$$\int_1^{x_{2n+1}}\frac{\varepsilon(t)}{t}\,dt=\int_1^{x_1}\frac{\varepsilon(t)}{t}\,dt-n+\sum_{k=1}^{2n}(-1)^{k-1}\alpha_k\to-\infty\ (n\to\infty),$$

$$\int_1^{x_{2n+2}}\frac{\varepsilon(t)}{t}\,dt=\int_1^{x_1}\frac{\varepsilon(t)}{t}\,dt+n+1+\sum_{k=1}^{2n+1}(-1)^{k-1}\alpha_k\to+\infty(n\to\infty).$$

Hence,

$$\underline{\lim}_{x\to+\infty}\int_1^{x}\frac{\varepsilon(t)}{t}\,dt=-\infty,\qquad \overline{\lim}_{x\to+\infty}\int_1^{x}\frac{\varepsilon(t)}{t}\,dt=+\infty,$$

that is,

$$\underline{\lim}_{x\to+\infty}L(x)=0,\qquad \overline{\lim}_{x\infty+\infty}L(x)=+\infty.$$

Remark. A particular example of a slowly varying function $L(x)$ with properties $\underline{\lim}_{x\to+\infty}L(x)=0$ and $\overline{\lim}_{x\to+\infty}L(x)=+\infty$ is given in [2] (p.58,ex.3), without detailed proof. Our result is evidently more general.

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REZIME

JEDNA PRIMEDBA KOJA SE ODNOSI NA SPORO PROMENLJIVE U SMISLU KARAMATE

Dat je kompletan dokaz jednog rezultata iz rada [1].

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THE NORMAL CONVERGENCE OF THE POWER SERIES IN THE n-DIMENSIONAL MIKUSINSKI DIFFERENTIATION OPERATOR

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ABSTRACT

Let $s^{(\alpha)} = s_1^{\alpha_1} \dots s_n^{\alpha_n}$, $(\alpha_i \geq 0,\ i=1,\dots,n)$ be the differentiation operator in the n-dimensional Mikusinski operational calculus and let $a_{(k)}$ be complex numbers depending on multi orders $(k) \in N_o^n$. The necessary and sufficient conditions for the normal convergence of power series

$$S = \sum_{(k)} a_{(k)} s^{(\alpha k)},\ ((\alpha k) = (\alpha_1 k_1, \dots, \alpha_n, k_n)).$$

in the space of n-dimensional Mikusinski operators are given. It is shown that the convergence depends on the quasi-analyticity of certain Lelong-Carleman class, which contains the factor of convergence.

This completes the results of T.Boehme, J.Wloka, B.Stanković and the author ([1],[6],[9]).

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Key words and phrases: Mikusinski operators, power series, quasi-analycity

1. INTRODUCTION

Our terminology and notation for the n-dimensional Mikusinski operators will be as in Gutterman's paper [3], (for n=2 as in Mikusinski's book [5]), and for quasi-analytic classes of functions of n variables as in Roumieu's paper [7].

We shall give the necessary and sufficient conditions for the normal convergence of a power series in the Mikusinski differentiation operator $s^{(\alpha)}$, $S = \sum_{(k)} a_{(k)} s^{(\alpha k)}$. S is convergent in $M(R^n)$ if a special class $M_{(k)} = M_{(k)}(a_{(k)})$ is not quasi-analytic (Theorem 4.1, Theorem 4.1.). Application of Lelong's theorem yields criteria in terms of the coefficients $a_{(k)}$, which are sufficient for S to be normally convergent (Theorem 4.2.).

2. n-DIMENSIONAL MIKUSINSKI OPERATORS

Let $C_o(R^n)$ denote the convolution ring of all continuous functions defined in R^n (n-dimensional Euclidean space) with the support in R^n_o, ($R^n_o = (x_1,\dots,x_n) \in R^n$, $x_i \geq 0$, $i=1,\dots,n$). The addition is the pointwise addition of functions and the convolution of $u(x_1,\dots,x_n)$ and $v(x_1,\dots,x_n)$ is the function $w(x_1,\dots,x_n)$ defined by the integral

$$w(x_1,\dots,x_n)=\int_o^{x_1}\dots\int_o^{x_n} u(x_1-t_1,\dots,x_n-t_n)v(t_1,\dots,t_n)dt_1\dots dt_n.$$

$C_o(R^n)$ has no divisors of zero [2],[3].

The field $M(R^n)$ of n-dimensional Mikusinski operators is the quotient field of $C_o(R^n)$. For an operator $a \in M(R^n)$, we shall use the formal notation of a quotient (the inverse operation to convolution) $a = \frac{u}{v}$, $u, v \in C_o(R^n)$ and $v(x_1,\dots,x_n) \neq 0$. Obviously, $\frac{u}{v}$ denotes the equivalence class a. To every function $u(x_1,\dots,x_n) \in C_o(R^n)$ there corresponds an operator. Thus, the set of operators contains that of functions. Further, we shall write a function $u(x_1,\dots,x_n)$ in the form u or $\{u(x_1,\dots,x_n)\}$. By $\{u(x_1,\dots,x_n)\}\{v(x_1,\dots,x_n)\}$, we denote the convolution and by $\{u(x_1,\dots,x_n)v(x_1,\dots,x_n)\}$ the ordinary product of two functions $u(x_1,\dots,x_n)$ and $v(x_1,\dots,x_n)$.

We define the operator ℓ as function $\{1\}$, $\ell=\{1\}$, and the operator ℓ_i^α by $\ell_i^\alpha = \frac{\{\frac{x_i^\alpha}{\Gamma(\alpha+1)}\}}{\ell}$, $\alpha>0$, $i=1,\dots,n$.

Let c be an arbitrary constant function. We define the numerical operator $[c]$ by $[c]=\frac{\{c\}}{\ell}$. Accordingly, the operator $I=[1]=\frac{\{1\}}{\ell}$ is the unity operator.

The inverse operators of ℓ and ℓ_i^α, $a>0$, $i=1,\dots,n$, are denoted, respectively, by s and s_i^α, and are referred to as differentiation operators.

In Lemma 1. of ([3], p.473) Gutterman proved that $\ell=\ell_1\dots\ell_n$ and $s=s_1\dots s_n$.

For the differentiation operators s_i and the differentiable function $u(x_1,\dots,x_n)$, we have the formula ([3], p.472):

$$s_i\{u(x_1,\dots,x_n)\}=\{u_{x_i}(x_1,\dots,x_n)\}+$$
$$+\ s_i\{u(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)\},$$

and in general for $|r|=r_1+\dots+r_n$, $r_i\geq 0$, $i=1,\dots,n$,

$$D^{(r)}u = \frac{\partial^{|r|}u}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} = s_1^{r_1}\dots s_n^{r_n}\{u(x_1,\dots,x_n)\}-d$$

where

$$d=\left(\sum_{k_1=0}^{r_i}\dots\sum_{k_n=0}^{r_n} s_1^{k_1}\dots s_n^{k_n}\, a^{|r|}_{(k_1,\dots,k_n)}-a^{|r|}_{(0,\dots,0)}\right), \tag{2.1}$$

$$a^{|r|}_{(k_1,\dots,k_n)}=\left\{\frac{\partial^{|r-k|}u(x_1,\dots,x_n)}{x_1^{r_1-k_1}\dots x_n^{r_n-k_n}}\right\}$$

at the point $(x_1\delta_0^{k_1},\dots,x_n\delta_0^{k_n})$, ($\delta_i^j$ is Kronecker's delta).

In Theorem 1. of ([3], p.474) Gutterman gives the conditions for an operator to be a function. The only function represented by operator (2.1.) is equal to zero.

In $M(R^n)$ we use the convergence of a sequence in the sense of the first Mikusinski convergence.

DEFINITION 2.1. *A sequence of operators* ω_k, $k=1,2,\dots$, *converges to the operator* ω, *if there exist a function* $g(x_1,\dots,x_n)\neq 0$ *from* $C_0(R^n)$ *and a sequence of functions* $g_k(x_1,\dots,x_n)\in C_0(R^n)$, $k=1,2,\dots$, *such that*

(i) $\omega g = f,\quad f(x_1,\dots,x_n)\in C_0(R^n)$

(ii) $\omega_k g = g_k,\quad k=1,2,\dots$

(iii) *The sequence* $g_k(x_1,\dots,x_n)$, $k=1,2,\dots$, *converges uniformly on every finite n-dimensional interval* $I_T=[0,T_1]\times\dots\times[0,T_n]$ *to* $f(x_1,\dots,x_n)$.

For the series $\sum_{(k)}\omega_{(k)}$, where $\omega_{(k)}$ are operators which depend on multi order, we shall consider the normal convergence.

DEFINITION 2.2. *A series* $\sum_{(k)}\omega_{(k)}$ *of n-dimensional operators depending on multi orders converges normally to the operator* ω, *if there exist a function* $g(x_1,\dots,x_n)\neq 0$ *from* $C_0(R^n)$ *and a sequence of functions* $g_{(k)}(x\ ,\dots,x\ )\in C_0(R^n)$, $(k)\in N_0^n$, *such that*

(i) $\omega_{(k)} g = g_{(k)},\quad (k)\in N_0^n$

(ii) $\sum_{(k)}\max_{I_T}|g_{(k)}(x_1,\dots,x_n)|$ *converges for every finite n-dimensional interval* $I_T=[0,T_1]\times\dots\times[0,T_n]$.

Obviously, condition (ii) implies that $\sum_{(k)} g_{(k)}$ exists and is independent of the order of summation. The sum $u(x_1,\ldots,x_n) = \sum_{(k)} g_{(k)}(x_1,\ldots,x_n)$ is in $C_o(R^n)$ and $\omega = \frac{u}{g}$.

3. QUASI-ANALYTIC CLASS

Let $M_{(p)}$ be a sequence of positive real numbers depending on multi orders $(p)\in N_o^n$ and K a regular compuct set in R^n. We always suppose $M_{(o)}=1$, $0<M_{(p)}\leq\infty$ for each (p), $M_{(p)}<\infty$ for infinitely many (p).

By $\varepsilon(K,M_{(p)})$, we mean the class of all the infinitely-differentiable functions on K, such that there are constants $\beta_f>0$, and h_f depending on f and

$$(3.1.)\qquad \max_{x\in K} |D^{(p)}f| \leq \beta_f h_f^{|p|} M_{(p)}$$

for each $(p)\in N_o^n$, $(p)=(p_1,\ldots,p_n)$, $|p|=p_1+\ldots+p_n$,

$$D^{(p)}f = \frac{\partial^{|p|}f}{\partial x_1^{p_1}\ldots\partial x_n^{p_n}} .$$

$\varepsilon(K,M_{(p)})$ is a vector space under the pointwise addition of functions. If K_1 and K are compact sets in R^n and (a), (b) n-tuples of real numbers such that $(x)\in K_1$ implies $(ax+b)^{(*)}\in K$, then for f in $\in\varepsilon(K,M_{(p)})$ we have $\psi=f(ax+b)\in\in(K_1,M_{(p)})$.

For an open set $\Omega\subseteq R^n$, by $\varepsilon(\Omega,M_{(p)})$ we mean the class of all functions f, such that $f\in\varepsilon(K,M_p)$ for each compact set $K\subset\Omega$.

$\mathcal{D}(\Omega,M_{(p)})$ is the set of all $f\in\varepsilon(\Omega,M_{(p)})$ which have a compact support.

A sequence $M_{(p)}$ is said to be logarithmically convex if

(*) $(ax+b)=(a_1x_1+b_1,\ldots,a_nx_n+b_n)$

$$(3.2) \qquad M^2_{(p)} \leq M_{(p-q)} M_{(p+q)}, \text{ for each } (p),\ (q) \in N^n_o.$$

In Theorem 4. of ([7], p.158) Roumieu proved the following:

THEOREM 3.1. *If there exist constants* A *and* H *such that*

$$(3.3.) \qquad M_{(p)} M_{(q)} \leq AH^{|p+q|} M_{(p+q)}, \text{ for each } (p),\ (q) \in N^n_o,$$

the vector space $\varepsilon(\Omega, M_{(p)})$ *is an algebra under the pointwise multiplication of functions.*

REMARK. If n=1, condition (3.3.) follows from (3.2.), but for $n \geq 2$ this is not always true ([7], p.159).

DEFINITION 3.1. *A class* $M_{(p)}$ *is said to be quasi-analytic, if* $\mathcal{D}(R^n, M_{(p)}) = \{0\}$.

For n=1, a necessary and sufficient condition for the quasi-analyticity of a class M_p is known.

THEOREM 3.2. ([8], p.375) *Let* M_p *be a logarithmically convex sequence. The class* M_p *is quasi-analytic if and only if* $f \in \varepsilon(R, M_p)$, $x_o \in R$ *and* $D^p f(x_o) = 0$ *for each* $p = 0, 1, \ldots$, *implies* $f(x) \equiv 0$ *on* R.

In the proof, we shall use that $\varepsilon(R, M_p)$ is an algebra under the pointwise multiplication of functions. In view of the remark in case n>1, we must suppose that the sequence $M_{(p)}$ satisfies (3.3.).

Now, we shall prove a similar characterization for the quasi-analyticity of a class $M_{(p)}$, for $n \geq 1$.

THEOREM 3.3. *Let the sequence* $M_{(p)}$ *satisfy inequality (3.3.). The class* $M_{(p)}$ *is not quasi-analytic if and only if there exist a function* $\psi(x_1, \ldots, x_n) \not\equiv 0$, $\psi \in \varepsilon(R^n, M_{(p)})$ *and some point* $(a_1, \ldots, a_n) \in R^n$ *such that*

$D^{(p)}\psi(x)=0$ *for each* $(x)=(x_1,\dots,x_n)$ *with* $x_i=a_i$ *for some* $i=1,\dots,n$, *and each* $(p)\in N_o^n$.

PROOF. If the class $M_{(p)}$ is not quasi-analytic, then by Definition 3.1, $\mathcal{D}(R^n,M_{(p)})$ contains a nontrivial function which evidently satisfies the hypotheses of the theorem.

Conversely, let $\psi(\bar{x})\neq 0$. We may assume $(\bar{x})>(a)$. If $g(x)=\psi(x)$ for $(x)\geq(a)$ and $g(x)=0$ otherwise, then $g(x)\in\varepsilon(R^n, M_{(p)})$.

Put $h(x)=g(x+a)g(2\bar{x}-a-x)$. By Theorem 3.1. $h(x)\in\mathcal{D}(R^n, M_{(p)})$ $h(\bar{x}-a)=\psi^2(\bar{x})\neq 0$ and $h(x)$ has a compact support. Thus $h(x)$ is a nontrivial member of $\mathcal{D}(R^n,M_{(p)})$.

There is a simple characterization due to Lelong, of the quasi-analytic classes in terms of the sequences M_p.

THEOREM 3.4. (Lelong, Theorem 1. [7], p.155.) *Let* M_p *be the rectified sequence of the sequence* $\inf_{|p|=p} M_{(p)}$ *in the sense of ([7], p.154.).*
The class $M_{(p)}$ *is not quasi-analytic if and only if*

$$\sum_{p=1}^{\infty}\frac{M_{p-1}}{M_p}<\infty$$

or

$$\sum_{p=1}^{\infty}(M_p)^{-\frac{1}{p}}<\infty.$$

4. THE CONVERGENCE OF POWER SERIES IN THE OPERATOR $s^{(\alpha)}$

The normal convergence of the power series

$$(4.1.)\qquad S=\sum_{(k)} a_{(k)}s^{(\alpha k)}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)\in R_o^n$, $a_{(k)}$ are complex numbers depending on multi orders (k) N_o^n and $s^{(\alpha k)}=s_1^{\alpha_1 k_1}\dots s_n^{\alpha_n k_n}$, can be characterized in terms of the quasi analyticity of Lelong-Carleman class.

Put

$$C_{(p)} = \begin{cases} \max\limits_{([\alpha k])=(p)} |a_{(k)}| & \text{if such } a_{(k)} \text{ exists}, (p) \in N_o^n \\ 0 & \text{otherwise}, \end{cases}$$

$([\alpha k])=([\alpha_1 k_1],\dots,[\alpha_n k_n])$, $[y]$ denotes the biggest integer $\leq y$.

THEOREM 4.1. *If the class* $C^{-1}_{(p)}$ *is not quasi-analytic, the series (4.1.) is normally convergent.*

THEOREM 4.1*. *If the sequence* $C^{-1}_{(p)}$ *satisfies (3.3.) and* S *is normally convergent in* $M(R^n)$, *then the class* $C^{-1}_{(p)}$ *is not quasi-analytic.*

PROOF of Theorem 4.1. Let $I_{T^*}=[0,T_1^*]\times\dots[0,T_n^*]$ be some finite interval. Since the class $C^{-1}_{(p)}$ is not quasi-analytic by Corollary 1. [7], p.156, there is a nontrivial non-negative function $\phi(x)\in \mathcal{D}(R^n,C^{-1}_{(p)})$ with the support in the interior of I_T^*. If h_ϕ is a constant, such that (3.1.) holds and $f(x)=\phi(\frac{x}{2h_\phi})$ for $\frac{x}{2h_\phi}\in I_T^*$, $f(x)=0$ otherwise, then $\ell^2 f\neq 0$ and by Theorem 1. of [3], $s^{(\alpha k)} f\in C_o(R^n)$ for each $(k)\in N_o^n$ and $(\alpha)\geq(0)$.

For each compact interval I_T there is some constant C such that

$$\max_{x\in I_T}|a_{(k)} s^{(\alpha k)}\ell^2 f| \leq \max_{x\in I_T^*} |a_{(k)}\ell^{([\alpha k])-(\alpha k)}\ell^2 D^{([\alpha k])}\phi x$$

$$\times \frac{1}{(2h_\phi)^{|([\alpha k])|}} \leq C \frac{1}{2^{|[\alpha k]|}}$$

$(|([\alpha k])|=[\alpha_1 k_1]+\dots+[\alpha_n k_n])$. Thus $\sum\limits_{(k)} a_{(k)} s^{(\alpha k)}$ is normally convergent in $M(R^n)$.

PROOF of Theorem 4.1*. If S is normally convergent in $M(R^n)$, there is a factor of convergence $g\in C_o(R^n)$ which must be a nontrivial infinitely differentiable function such that $D^{(p)}g(x)=0$ for every $(x)=(x_1,\dots,x_n)$ with $x_i=0$ for some $i=1,\dots,n$ (Theorem 1. of [3] and Definition 2.2.). Let I_T be some compact interval which contains the support number of g. By condition (ii) of Definition 2.2, there is a constant β_g such that

$$(4.3.)\qquad \max_{x\in I_T}|a_{(k)}s^{(\alpha k)}g|\leq\beta_g,\quad (k)\in N_o^n.$$

The function $\psi=\ell g$ satisfies (3.1.). Thus if $(p)=([\alpha k])$ for some $(k)\in N_o^n$ we have

$$\max_{x\in I_T}|D^{(p)}\ell g| = \max_{I_T}|\ell^{(k)-(p)}s^{(k)}\ell g|\leq\gamma C^{-1}_{(p)},$$

where γ is a constant. For other $(p)\in N_o^n$, (3.1.) is obviously satisfied. Therefore, the function $\psi=\ell g\in\varepsilon(R^n,C^{-1}_{(p)})$. The function ψ and the sequence $C^{-1}_{(p)}$ satisfy the requirements of Theorem 3., which proves that the class $C^{-1}_{(p)}$ is not quasi-analytic.

From Theorem 4.1. and Lelong's Theorem 3.4, we have another simple criteria, in terms of coefficients $a_{(k)}$, for S to be normally convergent.

THEOREM 4.2. *Let* $C_{(p)}$, $(p)\in N_o^n$ *be given by (4.2.). If* M_P *is the rectified sequence of the sequence* $\inf_{|p|=P} C^{-1}_{(p)}$ *in the sense of ([7], p.154.), the series* S *is normally convergent if one of the following conditions is satisfied:*

$$\text{a)}\quad \sum_{P=1}^{\infty}\frac{M_{P-1}}{M_P}\,,\qquad \text{B)}\quad \sum_{P=1}^{\infty}M_P^{-\frac{1}{P}}$$

REMARK. Conversely, if the sequence $C^{-1}_{(p)}$ satisfies (3.3.) and S is normally convergent, then, by Theroem 4.1. and Theorem 3.4, conditions A and B are satisfied.

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REZIME

NORMALNA KONVERGENCIJA STEPENIH REDOVA n-DIMENZIONALNOG OPERATORA DIFERENCIRANJA MIKUSINSKOG

Pitanje konvergencije operatorskog reda $\sum_{(k)} a_{(k)} s^{(\alpha k)}$ čiji članovi zavise od uredjenih n-torki, $a_{(k)}$ su kompleksni brojevi a $s^{(\alpha k)}$ je n-dimenzionalni operator diferenciranja Mikusinskog, dovedeno je u vezu sa osobinom ne kvazi-analitičnosti klase Carleman-Lelonga. Ovo je prirodna veza jer se u toj klasi u slučaju konvergencije nalazi funkcija koja predstavlja takozvani faktor konvergencije. Dati su dovoljni uslovi za konvergenciju Teorema 4.1. i Teorema 4.2, kao i potreban i dovoljan uslov za konvergenciju uz dodatno ograničenje (3.3.) Teorema 4.1.* Ovo ograničenje je prirodno u slučaju

dimenzije n>1, jer (3.3) nije više posledica logaritamske konveksnosti niza kao u slučaju n=1. Data je i jedna karakterizacija ne kvazi-analitičke klase u više dimenzionalnom slučaju (Teorema 3.3.) koja je pogodna za ovu problematiku operatorskog računa.

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MATHEMATICS SERIES

A NOTE ON α-REGULARITY AND COMPACTNESS

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ABSTRACT:

The aim of the paper is to study some properties of α-nearly compact (nearly compact), α-almost compact (almost compact) and compact sets (spaces) in topological spaces which are not Hausdorff or regular (almost regular). Some results concerning compactnessu are generalized.

1. PRELIMINARIES

Throughout the present paper, spaces will always mean topological spaces on which no sparation axioms are unless explicitly stated.

A subset of a spece X is *regularly open (regularly closed)* iff it is an interior (closure) of some closed (open) set or equivalently iff it is an interior (clo-

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sure) of its own closure (interior), [1] . A space X is *nearly compact* iff every regularly open cover has a finite subcovering, [5].

A space X is *almost compact* iff for every open cover $\{U_i : i\in I\}$ there exists a finite subfamily I_o of I such that $X=\cup\{\overline{U}_i : i\in I_o\}$, [4].

A subset A of a space X is said to be α-*nearly compact* (N-closed) iff for every regularly open cover $\mathcal{U}=\{U_i : i\in I\}$ of A there exists a finite subset I_o of I such that $A\subset\cup\ \{U_i : i\in I_o\}$, [6].

A subset A of a space X is said to be α-*almost compact* (H-closed) iff for every open cover $\mathcal{U}=\{U_i : i\in I\}$ of A there exists a finite subset I_o of I such that $A\subset\cup\{\overline{U}_i : i\in I_o\}$, [4].

A subset A of a space X is α-*Hausdorff* iff any two points a,b of a space X, where $a\in A$ and $b\in X\setminus A$, there are disjoint open sets U and V containing a and b respectively, [3].

A subset A of a space X is α-*regular* (α-*almost regular)* iff for any ponit $a\in A$ and any open (regularly open) set U containing a there exists an open set V such that $a\in V\subset\overline{V}\subset U$ or equivalently, for any closed (regularly closed) set F and any point $x\in A$ such that $x\in X\setminus F$, there exist disjoint open neighbourhoods of x and F respectively, [3], ([2]).

A space X is *almost regular* iff for any regularly closed set F and any point $x\notin F$, there exist disjoint open sets containing F and x respectively, [7].

2. SOME RESULTS

Theorem 2.1. If A is an α-regular and compact subset of a space X, then $\overline{A}$ is compact.

Proof. Let

$$\mathcal{U}=\{U_i : i\in I\}$$

be any open covering of A. For each $x\in A$, there exists an open set U_x containing x. There exists an open set V_x such that

$$x\subset V_x\subset V_x\subset U_x .$$

Let

$$\mathcal{V}=\{V_x : x\in A\}.$$

There exists a finite points $x_1, x_2, \ldots, x_n$ in A such that

$$A\in \bigcup_{i=1}^{n} V_{x_i}$$

Thus we have

$$\overline{A}\subset \bigcup_{i=1}^{n} V_{x_i} = \bigcup_{i=1}^{n} V_{x_i} \subset \bigcup_{i=1}^{n} U_{x_i} .$$

Hence $\overline{A}$ is compact.

Theorem 2.2. If A is an α-almost regular and α-nearly compact subset of a space X, then $\overline{A}$ is α-nearly compact.

Proof. It is similar to the proof of Theorem 2.1.

The closure of an α-regular and compact (α-almost regular and α-nearly compact) subset is not always α-regular (α-almost regular). The following example serves the purpose.

Example 2.1. Let

$$X=\{a,b,c,a_i : i=1,2,\ldots\}$$

Let

$$A=\{b,a_i : i=1,2,\ldots\}$$

Let each point a_i be isolated. Let a point c be isolated. Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a): n=1,2,\dots\},$$

where

$$V^n(a)=\{a,a_i: i\geqslant n\}.$$

Let the fundamental system of neighbourhoods of b be the set

$$\{U^n(b): n=1,2,\dots\},$$

where

$$U^n(b)=V^n(a)\cup\{b,c\}.$$

The set A is α-regular and compact.

$$\overline{A}=\{a,b,a_i: i=1,2,\dots\}$$

is compact, but is not α-almost regular (hence $\overline{A}$ is not α- regular) (for any regularly open neighbourhood $V^n(a)$ of a $\overline{V^n(a)}=V^n(a)\cup\{b\}$).

The closure of a regular and compact subset is not always compact, as we can see from the following example.

Example 2.2. Let

$$X=\{a,a_i: i=1,2,\dots\}$$

Let a point a be isolated. For each a_i let the fundamental system of neighbourhoods be the set

$$\{a,a_i\}.$$

The set

$$A=\{a\}$$

is regular and compact, but $\overline{A}=x$ is not compact (A is not α-regular).

The following example shows that there is a dense α-regular (α-almost regular) and compact (α-nearly compact) subset of a space X, which is not regular (almost regular)

Example 2.3. Let X be a space in example 2.1. Let

$$A=\{b,c,a_i:i=1,2,\dots\}$$

A is a dense α-regular and compact subset of a spece X. X is not almost regular (regular) at a, hence X is not almost regular (regular).

Theorem 2.3. Every α-regular α-almost compact is compact.

Proof. Let

$$\mathcal{U}=\{U_i:i\in I\}$$

be any open cover of an α-regular α-almost compact subset A. For each $x\in A$ there exists an $i(x)\in I$ such that $x\in U_{i(x)}$. There exists an open set V_x such that

$$x\in V_x\subset \overline{V}_x\subset U_{i(x)}.$$

Let

$$\mathcal{V}=\{V_x:x\in A\}.$$

Since $\mathcal{V}$ is an open cover of A, there exists a finite subfamily

$$\{V_{x_j}:j=1,2,\dots,n\}$$

such that

$$A\subset \bigcup_{j=1}^{n}\overline{V}_{x_j}.$$

Then,

$$\{U_{i(x_j)}:j=1,2,\dots,n\}$$

is a finite subcover of $\mathcal{U}$, hence A is compact.

Theorem 2.4. Every α-almost regular α-almost compact is α-nearly compact.

Proof. It is similar to the proof of Theorem 2.3.

Corollary 2.1. Every α-ragular α-nearly compact is compact.

Proof. Every α-nearly compact is α-almost compact.

Theorem 2.5. If in a space X there exists a dense α-regular (α-almost regular) compact (α-nearly compact), then X is compact (nearly compact).

Proof. The closure of α-regular compact (α-almost regular α-nearly compact) is compact (α-nearly compact).

Theorem 2.6. Let A be any dense α-regular subset of a space X such that every open cover of A is an open cover of X. X is almost compact iff it is compact.

Proof. If X is compact, then X is almost compact (every compact is almost compact). Let X be almost compact. Let

$$\mathcal{U}=\{U_i : i\in I\}$$

be any open covering of X. For each point $x\in A$ there exists $i(x)\in I$ such that $x\in U_{i(x)}$. Since A is α-regular, there exists an open set V_x such that

$$x\in V_x \subset \overline{V}_x \subset \mathcal{U}_{i(x)}.$$

Let

$$\mathcal{V}=\{V_x : x\in A\}.$$

$\mathcal{V}$ is an open cover of A i.e. of X. Since X is almost compact there exists a finite subfamily

$$\{V_{x_j} : j=1,2,\ldots,n\},$$

such that

$$X = \bigcup_{j=1}^{n} \bar{V}_{x_j}$$

Then,

$$\{U_{i(x_j)} : j=1,2,\ldots,n\}$$

is a finite subcovering of $\mathcal{U}$, hence X is compact. (A is compact).

Theorem 2.7. Let A be any dense α-almost regular subset of a space X such that every regularly open cover of A is a regularly open cover of X. X is almost compact iff it is nearly compact.

Proof. It is similar to the proof of Theorem 2.6. In Theorem 2.6. we assumed that every open cover of a dense α- regular subset is an open cover of X. This assumption cannot be dropped, as can be seen from the following example.

Example 2.4. Let

$$X=\{a_{ij}, a_i, a : i,j=1,2,\ldots\}$$

Let each point a_{ij} be isolated. Let the fundamental system of neighbourhoods of a_i be the set

$$\{U^n(a_i) : n=1,2,\ldots\},$$

where

$$U^n(a_i)=\{a_i, a_{ij} : j \geq n\}.$$

Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a):n=1,2,\ldots\},$$

where

$$V^n(a)=\{a,a_{ij}:i\geqslant n,j\geqslant n\}.$$

X is a Hausdorff space which is not regular at a point a, hence X is not compact (every Hausdorff compact is regular). X is almost compact.
Let

$$A=\{a_{ij},a_i\}\ .$$

A is an open α-regular subset of X. A is a dense subset. A is not compact ($\overline{A}$=X is not compact).
We know that every regular almost compact is compact.
Example 2.3. shows that there exists a space with properties as in Theorem 2.6., which is not regular.
This example shows that thare exists a pace with properties as in Theorem 2.7., which is not almost regular.

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REZIME

α - REGULARNOST I KOMPAKTABILNOST

U radu se ispituju neke osobine α-regularnih odnosno α-skoro regularnih skupova povezanih sa kompaktnošću. Dokazuje se ekvivalentnost skore kompaktnosti sa kompaktnošću (blizu kompaktnošću) u prostorima koji nisu regularni (skoro regularni).

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SOME PROPERTIES OF SUBSETS AND ALMOST CLOSED MAPPINGS

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ABSTRACT

In the paper some properties of α-Hausdorff subsets and almost closed mappings are studied.

1. INTRODUCTION

No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is *regularly open* iff IntClA=A. A subset A of a space X is *regularly closed* iff ClIntA = A, [9].

A subset A of a space X is α- *paracompact (α-nearly paracompact)* iff for every open (regularly open) cover

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$\mathcal{U}$ of A there is an open X-locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ and covers A, [13],[5].

A subset A of a space X is α-paracompact (*α-nearly paracompact) with respect to a subset* B iff for every open (regularly open) cover $\mathcal{U}=\{U_i: i\in I\}$ of A there exists an open family $\mathcal{V}=\{V_j: j\in J\}$ such that:

- $\mathcal{V}$ refines $\mathcal{U}$,
- $A \subset \cup\{V_j: j\in J\}$,
- $\mathcal{V}$ is locally finite at each point $x\in B$, [6].

Subsets A and B of a space X are *mutually α-paracompact (mutually α-nearly paracompact)* iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and the subset B is α-paracompact (α-nearly paracompact) with respect to the subset A, [6].

A subset A of a space X is *α-nearly compact or N-closed* iff every regularly open cover $\mathcal{U}=\{U_i: i\in I\}$ of A has a finite subcover of A, [1].

A space X is *nearly compact* iff every regularly open cover of has a finite subcover, [11].

A space X is *locally nearly compact* iff for each point $x\in X$, there exists an open neighbourhood U of x such that ClU is α-nearly compact, [1].

A subset A of a space X is *α-Hausdorff* iff for any two points a,b of a space X, where $a\in A$ and $b\in X\setminus A$, there are disjoint open sets U and V containing a and b respectively. A subset A of a space X is *α-regular (α-almost regular)* iff for any point $a\in A$ and any open (regularly open) subset U containing a there is an open subset V such that $a\in V\subset \mathrm{Cl}V\subset U$,[7];[4].

A space X is *almost regular* iff for any regularly closed set F and any point $x\notin F$, there are disjoint open sets containing F and x respectively, [10].

A mapping $f:X\to Y$ is *almost closed (almost open)* iff for any regularly closed (regularly open) set F of X,f(F) is closed (open) in Y, [9].

A mapping $f:X\to Y$ is *almost continuous* at a point $x\in X$ iff for every open neighbourhood M of f(x) there is an open neighbourhood N of X such that $f(N)\subset \mathrm{IntCl}M$. f is almost continuous iff it is almost continuous at each point of X, [9].

2. RESULTS

The following theorem was proved in [7]:

Theorem A. If A is an α-regular α-paracompact subset of a space X, then ClA is α-paracompact.
We can generalize this result with the following results:

Theorem 2.1. If A is an α-regular α-paracompact subset with respect to a subset B, then ClA is α-paracompact with respect to B.

Proof. It is similar to the proof of Theorem 2.4 in [7].

Theorem 2.2. If A is an α-almost regular α-nearly paracompact subset with respect to a subset B, then ClA is α-nearly paracompact with respect to B.

Proof. It is similar to the proof of Theorem 3.2 in [4].

Lemma 2.1. Let

$$\mathcal{U} = \{U_i : i\in I\}$$

be a family of open α-regular subsets of a space X such that:

a) $\mathcal{U}$ is locally finite at each point of a subset B

b) U_i is α-paracompact with respect to B, for each $i\in I$.

Then, $U=\cup\{U_i : i\in I\}$ is an open α-regular subset which is α-paracompact with respect to B.

Proof. By Lemma 2.1 in [2], the set U is α-regular. Let $\mathcal{V}=\{V_j : j\in J\}$ be an open covering of U. Then, $\{V_j\cap U_i : j\in J\}$ is an open covering of U_i, for each $i\in I$. Since U_i is α-paracompact with respect to B, there is a family $\mathcal{V}_i=\{D_k : k\in K^i\}$ of open sets such that:

- $\mathcal{V}_i$ refines $\{V_j\cap U_i : j\in J\}$,
- $U_i\subset\cup\{D_i : D_i\in\mathcal{V}_i\}$,
- $\mathcal{V}_i$ is locally finite at each point of B.

Consider the family

$$\mathcal{V}=\{D_k : k\in K^i, i\in I\}.$$

It follows that

- $\mathcal{V}$ refines $\mathcal{V}$,
- $U\subset\cup\{D : D\in\mathcal{V}\}$,
- $\mathcal{V}$ is locally finite at each point of B.

Thus, U is α-paracompact with respect to B.

Similarly, we can prove the next result:

Lemma 2.2. Let

$$\mathcal{U}=\{U_i : i\in I\}$$

be a family of regularly open α-almost regular subset of a space X such that:

a) $\mathcal{U}$ is locally finite at each point of a subset B,

b) for each $i \in I$, U_i is α-nearly paracompact with respect to B.

Then, $U = \cup\{U_i : i \in I\}$ is an open α-almost regular subset which is α-nearly paracompact with respect to B.

Theorem 2.3. Let

$\mathcal{U} = \{U_i : i \in I\}$

be a family of open α-regular subsets of a space X such that:

a) $\mathcal{U}$ is locally finite at each point of $X \setminus U \neq \emptyset$ $(U = \cup\{U_i : i \in I\})$,

b) U_i is α-paracompact with respect to $X \setminus U$, for each $i \in I$.

Then, U is on open - and - closed α-regular subset which is α-paracompact with respect to $X \setminus U$.

Proof. By Lemma 2.1, U is an open α-regular subset which is α-paracompact with respect to $X \setminus U$. By Theorem 2.6. in [6] , it follows that there is an open set V such that

$$U \subset V \subset \mathrm{Cl}V \subset U.$$

Thus $\mathrm{Cl}U = U$. Hence, the result.

In [12], Singal M.K. and Arya S.P. proved the next theorem:

Theorem B. Every nearly paracompact Hausdorff space is almost regular. In that theorem the Hausdorff property can be weakened as is shown by following result:

Theorem 2.4. Let X be a paracompact (nearly paracompact) space such that every closed (regularly closed) set is α-Hausdorff. Then X is regular (almost regular).

Proof. Let X be a paracompact (nearly paracompact) space and let F be any closed (regularly closed) subset of a space X and let $x \notin F$. Since every closed (regularly closed) subset of a paracompact (nearly paracompact) space is α-paracompact (α-nearly paracompact) and F is α-Hausdorff, it follows that there are open (regularly open) sets U and V such that

$$x \in U, F \subset V, U \cap V = \phi.$$

It follows that X is regular (almost regular).
Similarly, we have

Corollary 2.1. Let X be a compact (nearly compact) space such that every closed (regularly closed) subset is α-Hausdorff. Then, X is regular (almost regular).

Theorem 2.5. Let f $X \to Y$ be a closed almost continuous mapping of a space X onto a locally compact space Y such that for each $y \in Y$ $f^{-1}(y)$ is α-Hausdorff α-nearly compact. Then X is locally nearly compact.

Proof. By Theorem 2.3 in [4] Y is Hausdorff. Since Y is locally compact and Hausdorff it follows that, for each point $x \in X$ there is a closed compact neighbourhood V of f(x). Since f is almost continuous, the set $U=f^{-1}(\mathrm{Int}V)$ is open in X. By Theorem 1 in [8], the set $f^{-1}(V)$ is α-nearly compact in X. Since for each point $y \in Y, f^{-1}(y)$ is α-Hausdorff and the union of α-Hausdorff sets is α-Hausdorff, it follows that $f^{-1}(V)$ is α-Hausdorff. By Theorem 2.1 in [4] $f^{-1}(V)$ is closed. Now, we have

$$x \in U \subset \mathrm{Cl}U \subset f^{-1}(V).$$

Since every regularly closed subset of an α-nearly compact set is α-nearly compact, it follows that ClU is α-nearly compact. Now, U is an open neighbourhood of x such that ClU is α-nearly compact, hence X is locally nearly compact.

Corollary 2.2. ([8]) Let $f:X\to Y$ be a closed almost continuous surjection with N-closed point inverses. If X is Hausdorff and Y is locally compact, then X is locally nearly compact.

Theorem 2.6. Let f be an almost closed mapping of a space X onto a space Y. Let B be a closed subset of X such that for each $x\in X\setminus B$ the set $f^{-1}(f(x))$ is α-regular and α-paracompact with respect to B. Then, $f(B)$ is closed.

Proof. Let

$$y\in Y\setminus f(B).$$

Then

$$f^{-1}(y)\subset X\setminus B.$$

By Theorem 2.6 in [6],
there is an open neighbourhood of $f^{-1}(y)$ such that

$$f^{-1}(y)\subset V\subset \mathrm{Cl}V\subset X\setminus B.$$

Since f is almost closed, then there is an open set W in Y such that $y\in W$ and $f^{-1}(y)\subset f^{-1}(W)\subset \mathrm{IntCl}V\subset X\setminus B$. Thus, we have $y\in W\subset Y\setminus f(B)$. Hence the statement.

Theorem 2.7. Let X be an R_o space such that for each $x\in X$ $\mathrm{IntCl}(x)\neq\phi$. If $f:X\to Y$ is an almost closed mapping of the space X onto a space Y such that the family $\{f^{-1}(y):y\in Y\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact, then f is continuous.

Proof. Suppose that f is not continuous at some point $x\in X$. Let $\mathcal{U}(x)$ denote the family of all the open neighbourhoods of x. Let $y=f(x)$. Since f is not continuous at x, there is an open neighbourhood V of y such that

$$f(U)\cap(Y\setminus V)\neq\phi$$

for every $U\in\mathcal{U}(x)$. Thus,

$$\mathcal{A}=\{f(\mathrm{Cl}U)\cap(Y\setminus V):U\in\mathcal{U}(x)\}$$

is a family of closed subsets of Y such that

$$\cap\{f(\mathrm{Cl}U)\cap(Y\setminus V):U\in\mathcal{U}(x)\}\neq\phi$$

(X is R_o such that $\mathrm{IntCl}(x)\neq\phi$, for each $x\in X$. Thus, $U_o=\cap\{U:U\in\mathcal{U}(x)\}$ is an open set containing x and hence a member of $\mathcal{U}(x)$. So $(Y\setminus V)\cap f(U_o)\neq\phi$. i.e.

$$\cap\{A:A\in\mathcal{A}\}=f(\mathrm{Cl}U_o)\cap(Y\setminus V)\neq\phi).$$

Thus, there is a point $y_o\in\cap\{A:A\in\mathcal{A}\}$. Hence we have $y_o\in Y\setminus V$ and $x\notin f^{-1}(y_o)$. Since the family $\{f^{-1}(y):y\in Y\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact, there are disjoint regularly open sets U_x and U_o such that

$$x\in U_x \text{ and } f^{-1}(y_o)\subset U_o.$$

From

$$\mathrm{Cl}U_x\cap f^{-1}(y_o)\subset\mathrm{Cl}U_x\cap U_o=\phi$$

we have

$$y_o\notin f(\mathrm{Cl}U_x).$$

On the other hand, since U_x belongs to $\mathcal{U}(x)$, we have

$$y_o\in f(\mathrm{Cl}U_x)\cap(Y\setminus V)\subset f(\mathrm{Cl}U_x).$$

This is a contradiction. Hence, f must be continuous at x.

Thus, f is continuous.

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REZIME

NEKE OSOBINE PODSKUPOVA I SKORO ZATVORENIH PRESLIKAVANJA

U radu se ispituju neke osobine α-Hausdorfovih, α-regularnih i α-skoro regularnih podskupova topološkog prostora X. Daju se i uslovi kada je blizu parakompaktan prostor skoro regularan u prostoru koji ne mora da bude Hausdorfov. Daju se takodje i uslovi kada je skoro zatvoreno preslikavanje neprekidno nad prostorom koji ne mora da bude Hausdorfov.

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FIXED POINT THEOREMS IN TAKAHASHI CONVEX METRIC SPACES

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ABSTRACT

In this paper a generalization of a fixed point theorem from [7] is proved for a class of Takahashi convex metric spaces.

1. INTRODUCTION

In 1970 Takahashi [6] introduced the definition of convexity in a metric space and generalized some important fixed point theorems previously proved for Banach spaces. Subsequently, Machado [4], Talman [7], Gauy and Singh [1] , Hadžić and Gajić [2], Gajić [3], among others have obtained additional results in this setting. This paper is a continuation of these investigations.

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2. PRELIMINARIES

Definition 1. *Let* X *be a metric space and* I *be the closed unit interval. A mapping* $W: X \times X \times I \to X$ *is said to be a convex structure on* X *iff for all* $x, y \in X, \lambda \in I$,

$$d(u, W(x,y,\lambda)) \leqslant \lambda d(u,x) + (1-\lambda) d(u,y), \text{ for all } u \in X.$$

X *together with a convex structure is called a Takahashi convex metric space.*

Any convex subset of a Banach space is a Takahashi convex space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$.

Definition 2. *Let* X *be a convex metric space. A nonempty subset* K *of* X *is convex iff* $W(x,y,\lambda) \in K$ *whenever,* $x, y \in K$ *and* $\lambda \in I$.

Takahashi has shown that open and closed balls are convex and that the arbitrary intersection of convex sets is convex ([5]).

For arbitrary $C \subset X$ *let:*

$$\tilde{W}(C) := \{W(x,y,\lambda) : x, y \in C, \lambda \in [0,1]\}. \qquad (1)$$

It is easy to see that $\tilde{W}: \mathcal{P}(X) \to \mathcal{P}(X)$ *is a mapping with properties*

1. $C \subseteq \tilde{W}(C)$, *for* $C \in \mathcal{P}(X)$;
2. $C \subseteq B$ *implies* $\tilde{W}(C) \subseteq \tilde{W}(B)$, *for* $B, C \in \mathcal{P}(X)$;
3. $\tilde{W}(C \cap B) \subseteq \tilde{W}(C) \cap \tilde{W}(B)$, *for any* $B, C \in \mathcal{P}(X)$.

Using this notation we can say that $K \subseteq X$ *is convex iff* $\tilde{W}(K) \subseteq K$.

A few additional definitions and propositions will be needed subsequently.

Definition 3. *A convex metric space X is said to have Property (C) iff every decreasing net of nonempty, closed convex subsets of X has a nonempty intersection.*

Remark. Every weakly compact convex subset of a Banach space has Property (C).

Definition 4. *Let X be a convex metric space and A a nonempty closed convex bounded set in X. For $x \in X$ we set:*

$$r_x(A) = \sup_{y \in A} d(x,y),$$

$$r_x(A) = \inf_{x \in A} r_x(A),$$

$$A_c = \{x \in A : r_x(A) = r(A)\}, \text{ (centre of A);}$$

$$\delta(A) = \sup\{d(x,y) : x,y \in A\}; \text{ (diametar of A).}$$

Definition 5. *A point $x \in A$ is a diametral point of A iff* $\sup_{y \in A} d(x,y) = \delta(A)$.

Definition 6. *A convex metric space is said to have a normal structure iff for each closed bounded convex subset A of X, which contains at least two points, there exists $x \in A$ which is not a diametral point for A.*

Remark. Any compact convex metric space has a normal structure ([1]).

For sets $K, H \subset X$ by $\partial_F K$ we shall denote the boundary of K relative to H. If K is closed then:

$$\partial_H K=\{z\in K \mid B(z,r)\cap(H\setminus K)\neq\emptyset \text{ for each } r>0\}$$

$(B(z,r)=\{x\in X \mid d(x,z)<r\})$.

Lemma 1. [1] *Let* H *and* K *be two closed subset of the Takahashi convex metric space* X *such that* $H\cap K\neq\emptyset$

If $W:X\times X\times I\to X$ *is continuous in* $\lambda\in I$ *and* H *is convex, then* $\partial_H K=\emptyset \Leftrightarrow H\subseteq K$.

Lemma 2. *The continuous image of the convex subset* K *of a Takahashi convex metric space with a continuous convex structure* W *is a connected set.*

Proof. Since K is convex and W is continuous, K is pathwise connected, and since a continuous image of a pathwise connected set is pathwise connected, it is connected too.

Now, let us recall that the convex hull of a set A, $A\subset X$ is the intersection of all the convex sets in X containing A, and it is denoted by *conv* A.

It is obvious that if A is a convex subset of a convex metric space X, then

$$\tilde{W}^n(A)=\tilde{W}(W(\dots\tilde{W}(A))\dots)\subseteq A, \text{ for any } n\in N.$$

For $n\in N$ we set

$$A_n=\tilde{W}^n(A).$$

The sequence $\{A_n\}_{n\in N}$ is increasing so *lim inf* and *lim sup* exist and

$$\textit{lim sup}\ A_n=\textit{lim inf}\ A_n=\textit{lim}\ A_n=\bigcup_{n\in N}A_n\ .$$

Proposition 1. [3] *Let* X *be a Takahashi convex metric space.*
Then,

$$\mathit{conv}\ A = \lim A_n = \bigcup_{n\in N} A_n\,, \quad (A\subset X). \tag{2}$$

In the remainder of the paper, (X,d) will denote a complete metric space with a convex structure W.

Proposition 2. [3] *For any subset* A *of* (X,d)

$\delta(\mathit{conv}\ A)=\delta(A)$.

Lemma 3. *Let* $K\subset H$ *be two nonempty closed subsets in* (X,d), *let* H *be convex and let* W *be continuous in* λ. *If* $x\in K$ *and* $y\in H\setminus K$ *then there exists* $\lambda_o\in(0,1]$, *such that* $W(x,y,\lambda_o)\in\partial_H K$.

Proof. Let $L=\{\lambda\in[0,1]\,|\,W(x,y,\lambda)\in K\}$ since L is bounded and nonempty ($1\in L$), there exists *inf* L. Let $\lambda_o= \inf L$. If $\lambda_o\in(0,1]$, then for every $n\in N$ there exist λ_n, $\lambda_o\leq\lambda_n\leq 1$, $\lambda_n\to\lambda_o$ and so that $W(x,y,\lambda_n)\in K$. Since W is continuous $W(x,y,\lambda_o)\in K$, too. On the another hand, there exist $\{\tilde\lambda_n\}\, 0\leq\tilde\lambda_n<\lambda_o$, $\tilde\lambda_n\to\lambda_o$ so that $W(x,y,\tilde\lambda_n)\in H\setminus K$ and, then, $W(x,y,\tilde\lambda_n)\to W(x,y,\lambda_o)$ so we prove that $W(x,y,\lambda_o)\in\partial_H K$.

For $\lambda_o=0$, since $d(y,W(x,y,\lambda))=\lambda d(x,y)$, one can prove that $y\in\bar K=K$. Contradiction!

3. RESULTS

Theorem 1. *Let* (X,d) *be with a continuous convex structure and let* $H, K (K \subseteq H)$ *be nonempty closed convex subsets of* (X,d). *Further, let* H *be a normal subset and* K *a bounded set with Property* (C).

If $A:K\to H, A(\partial_H K)\subset K$, *and* A *is a mapping which, for all* $x,y\in K$ *satisfies the inequality:*

$$d(Ax,Ay)\leq ad(x,y)+b[d(x,Ax)+d(y,Ay)]+c[d(x,Ay) + + d(y,Ax)], \tag{3}$$

where a,b,c *are nonnegative constants such that* $a+2b+2c\leq \leq 1, a+b>0$, *then* A *has a fixed point.*

Poof. We shall assume that $a+2b+2c=1$. For fixed $\tilde{x}_o\in K$ and arbitrary $x\in K$ we have from (3):

$$d(Ax,A\tilde{x}_o)\leq ad(x,\tilde{x}_o)+b[d(x,Ax)+d(\tilde{x}_o,A\tilde{x}_o)] + +c[d(x,A\tilde{x}_o)+d(\tilde{x}_o,Ax)]\leq ad(x,\tilde{x}_o)+bd(x,\tilde{x}_o) + +bd(\tilde{x}_o,A\tilde{x}_o)+bd(A\tilde{x}_o,Ax)+bd(\tilde{x}_o,A\tilde{x}_o)+cd(x,\tilde{x}_o) + +cd(A\tilde{x}_o,\tilde{x}_o)+cd(\tilde{x}_o,A\tilde{x}_o)+cd(A\tilde{x}_o,Ax)$$

and further,

$$d(Ax,A\tilde{x}_o)\leq \frac{a+b+c}{1-b-c}\, d(x,\tilde{x}_o) + \frac{2(b+c)}{1-b-c}\, d(\tilde{x}_o,A\tilde{x}_o).$$

This and the boundness of K means that A is bounden on K.

Let, as in [7], $\mathcal{F}$ be the family of all the closed convex subsets of H so that for $F\in\mathcal{F}$, $F\cap K\neq \emptyset$ and

$A:F\cap K\to F$. Since $H\in\mathcal{F}$, $\mathcal{F}\neq\emptyset$. Let $\{F_\alpha\}$ be a decreasing chain of sets of $\mathcal{F}$ and let $F_0=\cap F_\alpha$. Note that $F_0\cap K$ is nonempty since $\{F_\alpha\cap K\}$ is a decreasing chain of a nonempty closed convex subset of a set with Property (C). Also, since $A:F_\alpha\cap K\to F_\alpha$, for each α, clearly $A:F_0\cap K\to F_0$. Since F_0 is a closed convex, $F_0\in\mathcal{F}$ so it follows by Zorn's Lemma that $\mathcal{F}$ has a minimal element.

Let F be a minimal element and suppose $\partial_F K\neq\emptyset$. We shall prove that $M\overset{\text{def}}{=\!=\!=}F\cap K$ has only one element. Suppose that M has more than one element. Then, center M_c is a nonempty closed convex set so that:

$$\delta(M_c)\leq r(M)<\delta(M).$$

Furthermore, if $\overline{conv}\,N$ denote the closed convex hull of set N, then we have

$$(\overline{conv}A(M))\cap K\supseteq A(M)\cap K\supseteq A(\partial_F K)\cap K=A(\partial_F K)\neq\emptyset$$

and

$$A(\overline{conv}A(M)\cap K)\subseteq A(F\cap K)=A(M)\subset\overline{conv}A(M).$$

From this two facts and from the minimality of the set F we get

$$\overline{conv}A(M)=F.$$

This and the boundedness of A give $\delta(F)<+\infty$.

Let $y\in M_c$. If $Ay\in M$ set $x_o=y$. If $Ay\notin M$, then $y\notin$

$\notin \partial_F K$ and by Lemma 3 there exists $0<\lambda_o<1$ such that $W(y,Ay,\lambda_o) \in \partial_F K$. In this case put $x_o=W(y,Ay,\lambda_o)$. In any case, we have $x_o, Ax_o \in M$ (since $A:\partial_F K\to K$) and that

$$r_0 \overset{\text{def}}{=\!=\!=} \sup\ \{d(x_0,z)\,|\,z\in M\} < \delta(F).$$

Further, for all $x\in M$ we have:

$$d(Ax,Ax_o)\leq ad(x,x_o)+bd(x_o,Ax_o)+bd(x,Ax) + \\ + cd(x_o,Ax)+cd(x,Ax_o)\leq (a+b)r_o+(b+2c)\delta(F),$$

so, since we prove that $F = \overline{conv}A(M)$,

$$r \overset{\text{def}}{=\!=\!=} \sup\ \{d(Ax_o,z)\,|\,z\in F\} < \delta(F). \tag{4}$$

Now, we shall define a transfinite sequence of the set $\{M_\alpha\}$ setting

$$M_0 = \{Ax_0\}$$

$$M_\alpha = \overline{conv}((M_{\alpha-1}\cap M)\cup A(M_{\alpha-1}\cap M)) \text{ if } \alpha-1 \text{ exists,}$$

$$M_\alpha = \overline{\bigcup_{\beta<\alpha} M_\beta} \qquad \text{if } \alpha-1 \text{ does not exists.}$$

Obviously, the sets M_α are nonempty convex (W is continuous) and closed subsets of the set F for which $M_\alpha \cap K\neq \emptyset$. For $\alpha<\alpha'$ it is $M_\alpha\subset M_{\alpha'}$. Taking an ordinal number α^* greater than the cardinal number of the power set of F, we see that in the sequence $\{M_\alpha\,|\,0\leq\alpha\leq\alpha^*\}$ there must be repetitions. If α_o is an ordinal number for which $M_{\alpha_0+1}=M_{\alpha_o}$ $(0\leq\alpha_o<\alpha^*)$, then we have

$$A(M_{\alpha_o} \cap M) \subseteq \overline{conv}((M_{\alpha_o} \cap M) \cup A(M_{\alpha_o} \cap M)) =$$

$$= M_{\alpha_o+1} = M_{\alpha_o}. \tag{5}$$

We shall prove for all $(0 \leqslant \alpha \leqslant \alpha_o)$ that

$$\delta(M_\alpha) \leqslant r. \tag{6}$$

For $\alpha=0$, (6) is true. Let $0<\alpha\leqslant\alpha_o$ and suppose that for all β, $0\leqslant\beta<\alpha$

$$\delta(M_\beta) \leqslant r \tag{7}$$

and

$$d(x,Ay) \leqslant r \quad \text{for} \quad x,y \in M_\beta \cap M. \tag{8}$$

Suppose that $\alpha-1$ exists. Taking a sequence $\{\varepsilon_n\}, \varepsilon_n>0, \varepsilon_n \to 0$ we can find $\tilde{x}_n, \tilde{y}_n \in M_\alpha$, so that

$$\delta(M_\alpha) - \varepsilon_n \leqslant d(\tilde{x}_n, \tilde{y}_n), \quad n=1,2,\ldots . \tag{9}$$

Further, we may assume that one of the following is true:

i) $\tilde{x}_n, \tilde{y}_n \in M_{\alpha-1} \cap M$, $n=1,2,\ldots$;

ii) $\tilde{x}_n = Ax_n, \tilde{y}_n = Ay_n, x_n, y_n \in M_{\alpha-1} \cap M$,

n=1,2,... ;

iii) $\tilde{x}_n \in M_{\alpha-1} \cap M, \tilde{y}_n = Ay_n, y_n \in M_{\alpha-1} \cap M$,

n=1,2,... .

If i) or iii) is true, then from (7), (8), (9) it follows immediately that $\delta(M_\alpha)\le r$. If ii) is true then (3), (7), (8) and (9) give $\delta(M_\alpha)\le r$, so inequality (7) is proved for $\alpha=\beta$.

Let $x\in M_\alpha\cap M$. Then for a $x\in M_\alpha\cap M$ and a given $\varepsilon>0$ we may take $x'\in$ *conv* $\{(M_{\alpha-1}\cap M)\cup A(M_{\alpha-1}\cap M)\}$ for which

$$d(x,Ax)<\varepsilon+d(x',Ax).$$

Using Proposition 1 we have that there exists $k_0\in N$, so that $x'\in\tilde{W}^{k_0}((M_{\alpha-1}\cap M)\cup A\cdot(M_{\alpha-1}\cap M))$. By (3), (7), (8) we get as in [7] that:

$$d(x,Ax)<\varepsilon+\sum_{i\in I_1}\omega_i d(u_i,Ax)+(a+b+c)(1-\omega)r+$$

$$+b(1-\omega)d(x,Ax)+c\sum_{i\in I_2}\omega_i d(u_i,Ax) \qquad (10)$$

$I_1\cup I_2=I$, card $I\le 2^{k_0}$, $\omega_i\ge 0$, $i\in I$, $\sum_{i\in I}\omega_i=1$

$\omega=\sum_{i\in I_1}\omega_i$ and $u_i\in M_{\alpha-1}\cap M$, for all $i\in I$.

This means that the set S of all β, $0\le\beta<\alpha$ for which

$$d(x,Ax)\le\varepsilon+\sum_{i\in I}\gamma_i d(v_i,Ax)+\gamma(a+b+c)r+$$

$$+\gamma b d(x,Ax) \qquad (11)$$

for some $v_i\in M_\beta\cap M$ and some real numbers $\gamma,\gamma_i\ge 0$ (I-finit set) for which

$$\gamma(a+2b+c)+\sum_{i\in I}\gamma_i=1$$

is nonempty. Let β_o be the first element in S. Suppose that $\beta_o>0$. If β_o-1 does not exist, then $v_i\in M_\beta, \beta<\beta_o$, because $M_{\beta_o}=\overline{\bigcup_{\beta<\beta_o} M_\beta}$. This is impossible because β_o is the first one in S. Hence β_o-1 exists and by definition

$$v_i\in conv\{(M_{\beta_o-1}\cap M)\cup A(M_{\beta_o-1}\cap M)\}.$$

Repeating for the distance $d(v_i,Ax)$ the above procedure, we may assume that in (11) $v_i\in M_{\beta_o}$ and that γ_i,γ are nonnegative constants and $\gamma(a+2b+c)+\Sigma\gamma_i=1$. This is a contradiction to the fact that β_o is the first element, so we have $\beta_o=0$.

Now, we can assume that in (11) $v_i\in M_0$ so

$$d(x,Ax)<\varepsilon+\left(\sum_{i\in I}\gamma_i\right)r+\gamma(a+b+c)r+$$

$$+\gamma b d(x,Ax)$$

and further,

$$d(x,Ax)<\frac{\varepsilon}{1-\gamma b}+\frac{\Sigma\gamma_i+\gamma(a+b+c)}{1-\gamma b}r$$

or since $0\leq\gamma\leq 1$

$$d(x,Ax)<\frac{\varepsilon}{1-b}+r.$$

Because $\varepsilon>0$ is arbitrary, we get

$$d(x,Ax)\leq r \tag{12}$$

Now, in a similar way, using (12) one can prove that (8) is true for $\beta=\alpha$.

In the case that $\alpha-1$ does not exist, from (7) for $\beta<\alpha$ it follows immediately that (7) holds for $\beta=\alpha$. Inequality (8), for $\beta=\alpha$, may be proved in a similar way as in the case when $\alpha-1$ exists, so we have that (7) and (8) are valid for all α, $0\leq\alpha\leq\alpha_o$.

Finally, we have that M_{α_o} is a nonempty closed convex subset of the set F with properties that

a) $\delta(M_{\alpha_o})\leq r<\delta(F)$

b) $M_{\alpha_o}\cap K=M_{\alpha_o}\cap M\neq\emptyset$

c) $A(M_{\alpha_o}\cap K)\subset M_{\alpha_o}$.

But this is a contradiction to the minimality of the set F, therefore $M=F\cap K$ has only one point.

Let $\{x^*\}=F\cap K$. From $\partial_F K\neq\emptyset$. We have $\{x^*\}=\partial_F K$. Then, $Ax^*\in F\cap K=x^*$ and x^* is the fixed point of A.

If $\partial_F K=\emptyset$ then, by Lemma 1, $F\subset K$ and $A:F\cap K\to F$ would imply $A:F\to F$. If F has more than one point, from the fact that H has a normal structure, one can see that there exist $x_o\in F$, for which

$$\sup\ \{d(x_o,y)\,|\,y\in F\}<\delta(F).$$

We can now construct the sequence $\{M_\alpha\}$, with $M_0=\{x_0\}$ and repeating the above procedure, with some simplifications, we again get a contradiction. Hence, F has only one element. This and $A:F\to F$ imply that A has a fixed point.

Theorem 2. *Let* (X,d) *be with a continuous convex structure,* H *and* K, $K \subset H$, *nonempty closed convex subset of* X. *Further, let* H *be a bounded normal subset and* K *with Property (C). If* $A: K \to H, A(\partial_H K) \subset K$ *and if* A *satisfies (3), where* $a,b,c \geq 0$ *and* $a+2b+2c \leq 1$, *then* A *has a fixed point.*

Proof. Let $\mathcal{F}$, F, M have the same meaning as in the proof of Theorem 1.

Suppose that $M=F \cap K$ has more than one element and $\partial_F K \neq \emptyset$. Since $F \subset H$ and H has a normal structure, there exists $x_0' \in F$ for which

$$r_0' = sup\{d(x_0', y) : y \in F\} < \delta(F).$$

Define $x_0 \in M$ in the following way :

1) if $x_0' \in M$ put $x_0 = x_0'$

2) if $x_0' \notin M$ take $x_0'' \in M \setminus \partial_F M$, chose $\lambda \in (0,1)$ such that $W(x_0'', x_0', \lambda) \in \partial_F M = \partial_F K$ and $x_0 = W(x_0'', x_0', \lambda)$

(Note that in the case $x_0' \notin M$ we have that $M \setminus \partial_F M \neq \emptyset$ because in the opposite case $\partial_F M = M$ and therefore $A: M \to M$. From the minimality of the set F, we now have M=F so $x_0' \in M$).

It is easy to see that in any case we have

$$sup\{d(x_0, y) \mid x \in F\} < \delta(F).$$

Now, we can construct the sequence $\{M_\alpha\}$, starting with $M_0 = \{x_0\}$ and so on.

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REZIME

TEOREME O NEPOKRETNOJ TAČKI U TAKAHAŠIJEVIM KONVEKSNIM METRIČKIM PROSTORIMA

U ovom radu dokazane su teoreme o nepokretnoj tački u klasi Takahašijevih konveksnih metričkih prostora koje uopštavaju rezultate za normirane prostore iz rada [7].

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THE SPACE OF THE GENERALIZED DIRICHLET SERIES OF SEVERAL COMPLEX VARIABLES ON A POLYDISC

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ABSTRACT

In this paper a version of the results, from the previous author's paper [8] for the space $X_D(f_1f_2)$ of the functions representable by the absolute and uniform convergent generalized Dirichlet series on a polydisc $D=D(R_1,R_2)$, is obtained. A representation of the continuous linear functional on $X_D(f_1,f_2)$ is constructed.

1. INTRODUCTION

We have investigated in paper [8] the space

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$X(f_1,f_2)$ of the absolute convergent generalized series (in the whole C^2)

$$(1.1) \qquad \sum_{m,n=0}^{\infty} a_{m,n} f_1(\lambda_m s_1) f_2(\mu_n s_2)$$

where f_1 and f_2 were entire functions of the given corresponding orders ρ_1 and types σ_1 and $\sigma_2, \lambda_0=\mu_0=0$. $(\lambda_m)_{m\geq 1}$, $(\mu_n)_{n\geq 1}$ are two sequences of complex numbers such that

$$0<|\lambda_1|<|\lambda_2|< \ldots \to \infty$$

and

$$0<|\mu_1|<|\mu_2|< \ldots \to \infty$$

and further

$$(1.2) \qquad \lim_{m+n\to\infty} \sup \frac{\ln(m+n)}{|\lambda_m|^{\rho_1}+|\mu_n|^{\rho_2}} = D<\infty.$$

In this way we have generalized some results by S.Daoud [1]-[3].

In this paper we shall investigate the generalized Dirichlet series (1.1) on a polydisc $D(R_1,R_2)=$ $=\{s_1:|s_1|<R_1\}\times\{s_2:|s_2|<R_2\}$ with a more restrictive condition on the sequences (λ_m) and (μ_n) than (1.2). Nemely, we shall suppose that

$$\lim_{m+n\to\infty} \frac{\ln(m+n)}{|\lambda_m|^{\rho_1} + |\mu_n|^{\rho_2}} = 0 \quad \text{holds.}$$

In this paper, for simplicity, we consider series and functions of only two variables, but all the results can easily be extended to any finite number of variables.
We shall construct a representation of the continuous linear functional on the space $X_D(f_1,f_2)$ of functions which have a representation with the generalized Dirichlet series.

2. Generalized Dirichlet series on a polydisc

The entire function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is of order ρ, $0<\rho<\infty$ and type $\sigma\neq 0,\infty$ if

$$\sup_{z\in\mathbb{C}} |f(z)| \exp(-(\sigma+\varepsilon)|z|^{\rho}) < +\infty$$

for all ε, $0<\varepsilon<1$.

Let $E_{\rho,\sigma}$ denote the set of all entire functions of order ρ and type σ such that $a_k\neq 0 (k=0,1,2,\ldots)$.

Let us consider the double generalized Dirichlet series

$$(2.1) \qquad F(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n} f_1(\lambda_m s_1) f_2(\mu_n s_2)$$

of complex variables s_1 and s_2, where $f_i \in E_{\rho_i,\sigma_i} (i=1,2)$. coefficients $a_{m,n}$ are complex numbers, $\lambda_0=\mu_0=0$, $(\lambda_m)_{m\geq 1}$, $(\mu_n)_{n\geq 1}$ are two sequences of complex numbers such that

$$0<|\lambda_1|\leq|\lambda_2|\leq \ldots \to \infty$$

and

$$0<|\mu_1|\leq|\mu_2|\leq \ldots \to \infty$$

and further

$$(2.2)\quad \lim_{m+n\to\infty} \frac{\ln(m+n)}{\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2}} = 0\ ,$$

where R_1 and R_2 are positive real numbers.

Theorem 2.1. Let

$$(2.3)\quad \lim_{m+n\to\infty} \sup \frac{\ln|a_{m,n}|}{\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2}} \leq -1.$$

Then, the series (2.1) converges absolutely and uniformly on the polydisc

$$D(R_1,R_2)=\{(s_1,s_2):|s_1|<R_1,|s_2|<R_2\}.$$

Proof. We take the real numbers r_o' , r_1 , r_o'' and r_2 such that $r_o' < r_1 < R_1$ and $r_o'' < r_2 < R_2$. By (2.3) there exist $N\in\mathbb{N}$, such that

$$\frac{\ln|a_{m,n}|}{\sigma_1 r_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 r_2^{\rho_2}|\mu_n|^{\rho_2}} < -1$$

for $m+n>N$. Hence,

$$(2.4)\quad |a_{m,n}|<\exp\{-\sigma_1 r_1^{\rho_1}|\lambda_m|^{\rho_1}-\sigma_2 r_2^{\rho_2}|\mu_n|^{\rho_2}\}$$

for $m+n>N$.

Since $f_i \in E_{\rho_i,\sigma_i}$, $(i=1,2)$, there exist natural numbers $N_1=N_1(\varepsilon)$ and $N_2=N_2(\varepsilon)$ for arbitrary ε, $0<\varepsilon<1$, such that

$$|f_1(\lambda_m s_1)|<k_1\exp\{|\lambda_m|^{\rho_1} r_o'^{\,\rho_1}(\sigma_1+\varepsilon)\}, \quad (|s_1|<r_o', r\geqslant N_1)$$

(2.5)

$$|f_2(\mu_n s_2)|<k_2 \exp\{|\mu_n|^{\rho_2} r_o''^{\,\rho_2}(\sigma_2+\varepsilon), \quad (|s_2|<r_o'', n\geqslant N_2)$$

for some

$$k_1>0, k_2>0.$$

Then, we obtain by (2.4) and (2.5)

(2.6) $$|a_{m,n}||f_1(\lambda_m s_1)||f_2(\mu_n s_2)|<k_1k_2 \exp\{-|\lambda_m|^{\rho_1}\cdot$$

$$\cdot[\sigma_1 r_1^{\rho_1}-(\sigma_1+\varepsilon)r_o'^{\,\rho_1}]-|\mu_n|^{\rho_2}[\sigma_2 r_2^{\rho_2}-(\sigma_2+\varepsilon)r_o''^{\,\rho_2}]\}$$

for $m+n\geqslant N_o=\max\{N,N_1+N_2\}$.

There exists such a positive number ε that the number u defined by

$$u := \min\left\{ \frac{\sigma_1 r_1^{\rho_1}-(\sigma_1+\varepsilon)r_o'^{\,\rho_1}}{\sigma_1 R_1^{\rho_1}}, \frac{\sigma_2 r_2^{\rho_2}-(\sigma_2+\varepsilon)r_o''^{\,\rho_2}}{\sigma_2 R_2^{\rho_2}} \right\}$$

is positive. Now, we can choose a positive number t such that $t<u$. By (2.2) there exists $N'\in N$, such that

$$t(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2})>\ln(m+n)$$

for $m+n\geqslant N'$. Hence, by (2.6)

$$|a_{m,n}||f_1(\lambda_m s_1)||f_2(\mu_n s_2)|<k_1k_2\ \exp\{-u(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2})\}$$

$$<k_1k_2\ \exp\{-\frac{u}{t}\ln(m+n)\}=\frac{k_1k_2}{(m+n)^{u/t}}$$

for $m+n>\max\{N_o,N'\}$ and $|s_1|\leq r_o'$, $|s_2|\leq r_o''$.

For special entire functions f_i, we have the following

Theorem 2.2. Let

$$f_i(z)=E_{\rho_i}(\sigma_i^{1/\rho_i}z),\ 0<\rho<\infty,\sigma\neq 0,\infty,i=1,2,$$ where E_ρ is the Mittag-Leffler function, i.e.

$$E_\rho(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\frac{n}{\rho}+1)}\ .$$

The series in (2.1) converges absolutely and uniformly on $D(R_1,R_2)$, iff (2.3) holds.

Proof. By Theorem 2.1. we have to prove only that if the series in (2.1) converges, then (2.3) holds. We suppose the contrary, i.e. that for some series

$$\sum_{m,n=0}^{\infty} a_{m,n}E_{\rho_1}(\sigma_1^{1/\rho_1}s_1)E_{\rho_2}(\sigma_2^{1/\rho_2}s_2)$$

the relation (2.3) does not hold, although the series is an absolute convergent for arbitrary $(s_1,s_2)\in D(R_1,R_2)$.
Let r_i $(i=1,2)$ be such numbers that $r_i<R_i$ holds. Since (2.3) does not hold for r_i', $0<r_i'<r_i$, there exist two subsequences $\{m_k\}$ and $\{n_k\}$, such that

$$\frac{\ln|a_{m_k n_k}|}{\sigma_1 {r'_1}^{\rho_1}|\lambda_{m_k}|^{\rho_1}+\sigma_2 {r'_2}^{\rho_2}|\lambda_{n_k}|^{\rho_2}} > -1 \quad (k\in\mathbb{N}).$$

Hence,

(2.7) $|a_{m_k,n_k}|<\exp\{-\sigma_1 {r'_1}^{\rho_1}-\sigma_2 {r'_2}^{\rho_2}|\mu_{n_k}|^{\rho_2}\}$:

We shall need the following asymptotic behaviour of the function E_ρ (see p. 134 in [4])

(2.8) $E_{\rho_i}(\sigma_i^{1/\rho_i}s_i)=\rho_i e^{\sigma_i s_i^{\rho_i}}+0(1/s_i),(i=1,2)$

in a small angle $|\arg s_i|<\psi_i$. There exist subsequences $\{s_k\}$ of $\{m_k\}$ and $\{u_k\}$ of $\{n_k\}$, such that $\arg \lambda_{s_k}\to\phi_1$ and $\arg \mu_{u_k}\to\phi_2$ as $k\to\infty$. By (2.8), for $s_i^0=r_i e^{-i\phi_i}$ $(i=1,2)$ for some small $\varepsilon>0$ and $k>k_0$

$$|E_{\rho_1}(\sigma_1^{1/\rho_1}\lambda_{s_k}S_2^0)|>\exp\{(\sigma_I-\varepsilon)r_I^{\rho_1}|\lambda_{s_k}|^{\rho_1}\}$$

(2.9)

$$|E_{\rho_2}(\sigma_2^{1/\rho_2}\mu_{u_k}s_2^0)|>\exp\{(\sigma_2-\varepsilon)r_2^{\rho_2}|\mu_{u_k}|^{\rho_2}\}.$$

So for $k>k_0$ (2.7) and (2.9) imply

$$|a_{s_k,u_k}||E_{\rho_1}(\sigma_1^{1/\rho_I}\lambda_{s_k}s_1^0)||E_{\rho_2}(\sigma_2^{1/\rho_2}\mu_{u_k}s_2^0)|$$
$$>\exp\{|\lambda_{s_k}|^{\rho_1}[\sigma_1 r_1^{\rho_1}-(\sigma_1+\varepsilon){r'_1}^{\rho_1}] -$$

$$-|\mu_n|^{\rho_2}[\sigma_2 r_2^{\rho_2}-(\sigma_2+\varepsilon)r_2''^{\rho_2}]\} \ .$$

We can choose a positive real number ε small enough so that the numbers

$$\sigma_1 r_1^{\rho_1}-(\sigma_1+\varepsilon)r_1'^{\rho_1} \quad \text{and} \quad \sigma_2 r_2^{\rho_2}-(\sigma_2+\varepsilon)r_2''^{\rho_2} \quad \text{are}$$

positive.

Hence,

$$|a_{s_k,u_k}||E_{\rho_1}(\sigma_1^{1/\rho_1}\lambda_{s_k}s_1^0)||E_{\rho_2}(\sigma_2^{1/\rho_2}\mu_{u_k}s_2^0)|>1$$

and the series in (2.1) does not converge for the point $(s_1^o,s_2^o)\in D(R_1,R_2)$. This contradiction implies that condition (2.3) must hold.

Using the idea of the preceding proof, we can prove the following theorem.

Theorem 2.3. Let

$$f_i(z)=E_{\rho_i}(\sigma_i^{1/\rho_i}z),\ 0<\rho<\infty\ .\ 0<\rho<\infty,\quad \sigma\neq 0,\infty, i=1,2.$$

If condition

$$(2.10)\quad \lim_{m+n\to\infty}\sup \frac{\ln|a_{m,n}|}{\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2}} = -1$$

holds for some positive real numbers R_1 and R_2, then for any pair (K_1,K_2) of real numbers such that $K_i>R_i\,(i=1,2)$, there exists a point $s^o=(s_1^o,s_2^o)\in C^2$ such that $R_i<|s_i^o|<K_i\,(i=1,2)$ and the series

$$(2.11)\qquad \sum_{m,n=0}^{\infty} a_{m,n}E_{\rho_1}(\sigma_1^{1/\rho_1}s_1^o)E_{\rho_2}(\sigma_2^{1/\rho_2}s_2^o)$$

does not converge.

Proof. Let K_i $(i=1,2)$ be positive real numbers such that $R_i<K_i\,(i=1,2)$ holds. We shall find a point $s^o=(s_1^o,s_2^o)$ such that the series (2.11) does not converge. Let $r_i\,(i=1,2)$ be real numbers such that $R_i<r_i<K_i$ $(i=1,2)$. By (2.10) there exist two subsequences $\{m_k\}$ and $\{n_k\}$ such that

$$\frac{\ln|a_{m_k n_k}|}{\sigma_1 r_1^{\rho_1}|\lambda\ |^{\rho_1}+\sigma_2 r_2^{\rho_2}|\mu_{n_k}|^{\rho_2}} > -1 \ (k\in \mathbb{N}).$$

Hence,

$$(2.12)\qquad |a_{m_k,n_k}|<\exp\{-\sigma_1 r_1^{\rho_1}|\lambda_{m_k}|^{\rho_1}-\sigma_2 r_2^{\rho_2}|\mu_{n_k}|^{\rho_2}\}\ .$$

There exist subsequences $\{s_k\}$ of $\{m_k\}$ and $\{u_k\}$ of $\{n_k\}$ such that $\arg\lambda_{s_k}\to\phi_1$ and $\arg\mu_{u_k}\to\phi_2$ as $k\to\infty$. By the asymptotic behaviour (2.8) of the Mittag-Leffler function $E_{\rho_i}(\sigma_i^{1/\rho_i}s_i)$, we have for $s_i^o=K_ie^{-i\phi_i}$ $(i=1,2)$ for some small $\varepsilon>0$ and $k>k_o$

$$|E_{\rho_1}(\sigma_1^{1/\rho_1}\lambda_{s_k}s_1^0)|>\exp\{(\sigma_1-\varepsilon)K_1^{\rho_1}|\lambda_k|^{\rho_1}$$

$$|E_{\rho_2}(\sigma_2^{1/\rho_2}\mu_{u_k}s_2^0)|>\exp\{(\sigma_2-\varepsilon)K_2^{\rho_2}|\mu_{u_k}|^{\rho_2}\}.$$

Hence, by (2.12),

$$|a_{s_k,u_k}||E_{\rho_1}(\sigma_1^{1/\rho_1}\lambda_{s_k}s_1^0)||E_{\rho_2}(\sigma_2^{1/\rho_2}\mu_{u_k}s_2^0)|$$

$$>\exp\{|\lambda_{s_k}|^{\rho_1}[\sigma_1 K_1^{\rho_1}-(\sigma_1+\varepsilon)r_1^{\rho_1}]$$

$$-|\mu_{u_k}|^{\rho_2}[\sigma_2 K_2^{\rho_2}-(\sigma_2+\varepsilon)r_2^{\rho_2}]\}.$$

We can choose a positive real number ε small enough so that the numbers

$\sigma_1 K_1^{\rho_1}-(\sigma_1+\varepsilon)r_1^{\rho_1}$ and , $\sigma_2 K_2^{\rho_2}-(\sigma_2+\varepsilon)r_2^{\rho_2}$ are positive.

positive.

Hence,

$$|a_{s_k,u_k}||E_{\rho_1}(\sigma_1^{1/\rho_1}\lambda_{s_k}s_1^0)||E_{\rho_2}(\sigma_2^{1/\rho_2}\mu_{u_k}s_2^0)|>1$$

and the series (2. 11), does not converge.

3. Linear continuous functionals

Let $X_D(f_1,f_2), f_i\in E_{\rho_i,\sigma_i}$ (i=1,2), denote the set of all the functions defined by the absolute and uniform convergent series from (2.1) on the polydisc $D(R_1,R_2)$, where the sequences $\{\lambda_m\}$ and $\{\mu_n\}$ satisfy condition (2.2) and $\{a_{m,n}\}$ satisfies the condition (2.3).

We can introduce a paranorm (see [9]) on $X_D(f_1,f_2)$ in the following way

$$\|F\| = \sup\{|a_{0,0}|, |a_{m,n}|^{(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2})^{-1}} \quad m+n\neq 0,\ m,n\geq 0\}$$

$$(F\in X_D(f_1,f_2)).$$

We can prove, in an analogous way as in [2], the following

Theorem 3.1. The set $X_D(f_1,f_2)$ endowed with the metric

$$d(F,G)=\|F - G\| \quad (F,G\in X_D(f_1,f_2)).$$

is a linear metric space.

We shall need the following version of Lemma 2 from [8]

Lemma 3.2. The following conditions are equivalent:

a)

$$\lim_{m+n\to\infty} \frac{\ln(m+n)}{\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2}} = 0;$$

b)

$$\lim_{m\to\infty} \frac{\ln m}{|\lambda_m|^{\rho_1}} = 0 ,$$

$$\lim_{n\to\infty} \frac{\ln n}{|\mu_n|^{\rho_2}} = ;$$

c) for any $\alpha, 0<\alpha<\infty$, the series

$$\sum_{m,n=0}^{\infty} \exp\{ -\alpha(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2})$$

converges.

Proof. The proof is analogous to the proof of Lemma 1 from [1] with some obvious modifications.

Theorem 3.3. Every continuous linear functional Φ on $X_D(f_1,f_2)$ is of the form

$$(3.1)\qquad \Phi(F) = \sum_{m,n=0} a_{m,n} c_{m,n}$$

$$F(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n} f_1(\lambda_m s_1) f_2(\mu_n s_2) \; ,$$

where

$$\ln|c_{o,o}|<M<1 \quad \text{and}$$

$$(3.2)\qquad \ln|c_{m,n}|^{(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2})^{-1}} < M<1$$

$(m+n\geq 0,\ m+n \neq 0)$.

Conversely, for each infinite matrix $[c_{m,n}]$ with property (3.2), (3.1) represents a continuous linear funcitional on $X_D(f_1,f_2)$.

Proof. Let Φ be a continuous linear functional on $X_D(f_1,f_2)$. We can take

$$\Phi(f_1(\lambda_m s_1) f_2(\mu_n s_2)) = c_{m,n} \quad (m,n \in \mathbb{N} \cup \{0\}).$$

Using the fact that the series representing the function F is uncoditionally convergent, we have

$$\Phi(F) = \lim_{m+n\to\infty} \Phi\Big(\sum_{i,j=0}^{m,n} a_{ij} f_1(\lambda_m s_1) f_2(\mu_n s_2) =$$

$$= \lim_{m+n\to\infty} \sum_{i,j=0}^{m,n} a_{i,j} c_{i,j} = \sum_{m,n=0}^{\infty} a_{m,n} c_{m,n} .$$

By Lemma 3.2 and (2.3) we have that for the matrix $[c_{m,n}]$ (3.2) holds.

Assume now that for the matrix $[c_{m,n}]$, (3.2) First we have to prove that the series

$$\sum_{m,n=0}^{\infty} a_{m,n} c_{m,n} \quad \text{converges.}$$

By (3.2) we have

(3.3) $|c_{o,o}| < \exp M$, $|c_{m,n}| < \exp\,(\sigma_1 R_1^{\rho_1} |\lambda_m|^{\rho_1} + \sigma_2 R_2^{\rho_2} |\mu_n|^{\rho_2})$.

By (2.3) there exists a natural number N such that

$$|a_{m,n}| < \exp\{ -\sigma_1 R_1^{\rho_1} |\lambda_m|^{\rho_1} - \sigma_2 R_2^{\rho} |\mu_n|^{\rho_2} \}$$

for $m+n \geq N$. Hence, by (3.3),

$$|a_{m,n} c_{m,n}| < \exp\{ -(1-M)(\sigma_1 R_1^{\rho_1} |\lambda_m|^{\rho_1} + \sigma_2 R_2^{\rho_2} |\mu_n|^{\rho_2})\}.$$

Since, by Lemma 3.2, the series

$$\sum_{m,n=0}^{\infty} \exp\{-(1-M)(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{\rho_2}|\mu_n|^{\rho_2})$$

converges, we obtain that the series

$$\sum_{m,n=0}^{\infty} a_{m,n}c_{m,n}$$

converges absolutely.

It is obvious that the functional Φ is linear. We have to prove that it is continuous. Let $\{F_k\}$ be a seguence from $X_D(f_1,f_2)$ which tends to zero with respect to the convergence in $X_D(f_1,f_2)$. We have to prove that $\lim_{k\to\infty}\Phi(F_k)=0$. Let

$$F_k(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}^{(k)} \; f_1(\lambda_m s_1)f_2(\mu_n s_2) \quad (k\in\mathbb{N}) .$$

The condition $F_k \xrightarrow{X_D(f_1,f_2)} 0$, as $k\to\infty$ implies that there exists a natural number k_o such that

$$|a_{o,o}^{(k)}|<\exp\{-k\}, |a_{m,n}^{(k)}|< \exp\{-k\;(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{2}|\mu_n|^{\rho_2})$$

for each k such that $k\geq k_o$. Hence, by (3.2),

$$|\Phi(F_k)|\leq \sum_{m,n=0}^{\infty} |a_{m,n}^{(k)}||\,c_{m,n}|<\exp\{M-k\}+$$

$$+ \sum_{m,n>0}^{\infty} \exp\{(M-k)(\sigma_1 R_1^{\rho_1}|\lambda_m|^{\rho_1}+\sigma_2 R_2^{2}|\mu_n|^{\rho_2})\}$$

for each k such that $k\geq k_o$. Letting $k\to\infty$, we obtain by Lemma 3.2 (since $M<1$) that $\Phi(F_k)\to 0$. This completes the proof of the theorem.

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REZIME

PROSTOR UOPŠTENIH DIRICHLETOVIH REDOVA OD VIŠE KOMPLEKSNIH PROMENLJIVIH NA POLI-DISKU

U radu se daje karakterizacija koeficijenata uopštenog Dirichletovog reda od više kompleksnih promenljivih, a koji konvergira apsolutno i uniformno nad poli-diskom $D(R_1,R_2)$. Daje se reprezentacija neprekidne linearne funkcionele nad prostorom $X_D(f_1,f_2)$, funkcija razvijenih u uopšteni Dirichlet red nad poli-diskom $D=D(R_1,R_2)$.

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ON TWO DEFINITIONS OF THE NON-COMMUTATIVE CONVOLUTION PRODUCT OF DISTRIBUTIONS

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ABSTRACT

Two generalizations of the standard definition of the convolution product of two generalized functions are given. It is shown that neither is a generalization of the other.

The standard way of defining the convolution product $f * g$ of certain pairs of distributions f and g is as follows, see for example Gel'fand and Shilov [7].

DEFINITION 1. Let f and g be distributions satisfying either of the following conditions:

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Then the convolution product $f * g$ is defined by

$$\langle (f * g)(x), \phi(x) \rangle = \langle g(y), \langle f(x), \phi(x + y) \rangle \rangle$$

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Key Words and Phrases: convolution product, distribution, neutrix, neutrix limit.

for arbitrary test function ϕ in the space $\mathcal{D}$ of infinitely differentiable functions with compact support.

This definition of the convolution product is very restrictive and in order that further convolution products could be defined, the next definition was introduced in [2].

DEFINITION 2. Let f and g be distributions and let τ be an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leq \tau(x) \leq 1$,

(iii) $\tau(x) = 1, \qquad |x| \leq 1/2$,

(iv) $\tau(x) = 0, \qquad |x| \geq 1$.

Let $f_n(x) = f(x)\tau(x/n)$ for $n = 1, 2, \ldots$. Then the convolution product $f * g$ is defined as the limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ in $\mathcal{D}$.

In this definition the convolution product $f_n * g$ is in the sense of Definition 1, the distribution f_n having bounded support.

It was proved in [2] that if a convolution product $f * g$ existed by Definition 1 then it also existed by Definition 2 and defined the same distribution. Definition 2 is therefore a generalization of Definition 1. Although the convolution product of Definition 1 is always commutative, the convolution product of Definition 2 does give some convolution products which are non-commutative.

The next definition also generalizes Definition 1 and was given in [3].

DEFINITION 3. Let f and g be distributions and let τ_n be the infinitely differentiable function defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

where τ is defined as in Definition 2. Let $f_n(x) = f(x)\tau_n(x)$ for $n = 1, 2, \ldots$. Then the convolution product $f * g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\underset{n\to\infty}{\mathrm{N-lim}}\langle f_n * g, \phi\rangle = \langle h, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \qquad \ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \ldots$, and all functions $\epsilon(n)$ for which $\lim_{n\to\infty} \epsilon(n) = 0$.

The convolution product $f_n * g$ in this definition is again in the sense of Definition 1, the support of f_n being contained in the interval $[-n - n^{-n}, n + n^{-n}]$.

It was shown in [3] that all the convolution products which were proved to exist by Definition 2 in [2] also existed by Definition 3. Examples were also given where the convolution product existed by Definition 3 but did not exist by Definition 2. Since then, in [4], [5] and [6], further examples were given where the convolution product existed by Definition 3 but not by Definition 2. All this suggests that every convolution product which exists by Definition 2 must also exist by Definition 3. This however is not the case and we now give an example where a convolution product exists by Definition 2 but not by Definition 3.

EXAMPLE. The convolution product

$$\sum_{k=1}^{\infty} 2^{-k}\delta'(x - k - \alpha k^{-k}) * \sum_{k=1}^{\infty} \delta(x + k + \alpha k^{-k}),$$

where $1/2 < \alpha < 1$, exists by Definition 2 but not by Definition 3.

PROOF. Put

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} 2^{-k}\delta'(x - k - \alpha k^{-k}), \\ g(x) &= \sum_{k=1}^{\infty} \delta(x + k + \alpha k^{-k}). \end{aligned}$$

Then with $n \geq 3$,

$$f_n(x) = \sum_{k=1}^{\infty} 2^{-k}\delta'(x - k - \alpha k^{-k})\tau(x/n)$$

$$\begin{aligned} &= \sum_{k=1}^{[n/2]} 2^{-k}\delta'(x-k-\alpha k^{-k}) \\ &\qquad + \sum_{k=[n/2]+1}^{n-1} 2^{-k}\tau\left(\frac{k+\alpha k^{-k}}{n}\right)\delta'(x-k-\alpha k^{-k}) \\ &\qquad - \sum_{k=[n/2]+1}^{n-1} 2^{-k}n^{-1}\tau'\left(\frac{k+\alpha k^{-k}}{n}\right)\delta(x-k-\alpha k^{-k}) \\ &= f_{1n}(x)+f_{2n}(x)-f_{3n}(x), \end{aligned}$$

where $[n/2]$ denotes the integer part of $n/2$.

The convolution products $f_{jn} * g, j = 1, 2, 3$, exist by Definition 1 and

$$\begin{aligned} f_{1n} * g &= \sum_{k=1}^{\infty}\sum_{i=1}^{[n/2]} 2^{-i}\delta'(x+k-i+\alpha k^{-k}-\alpha i^{-i}), \\ f_{2n} * g &= \sum_{k=1}^{\infty}\sum_{i=[n/2]+1}^{n-1} 2^{-i}\tau\left(\frac{i+\alpha i^{-i}}{n}\right)\delta'(x+k-i+\alpha k^{-k}-\alpha i^{-i}), \\ f_{3n} * g &= \sum_{k=1}^{\infty}\sum_{i=[n/2]+1}^{n-1} n^{-1}2^{-i}\tau'\left(\frac{i+\alpha i^{-i}}{n}\right)\delta(x+k-i+\alpha k^{-k}-\alpha i^{-i}). \end{aligned}$$

It is clear that the supports of these convolution products are contained in the unions of the open intervals $(r-\frac{1}{2}, r+\frac{1}{2})$ for $r = 0, \pm1, \pm2, \ldots$.

For a given r let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $(r-\frac{1}{2}, r+\frac{1}{2})$. If $r \geq 0$, then

$$\langle f_{1n} * g, \phi\rangle = -\sum_{k=r+1}^{r+[n/2]} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}),$$

and so

$$\lim_{n\to\infty}\langle f_{1n} * g, \phi\rangle = -\sum_{k=r+1}^{\infty} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}).$$

If $r < 0$, then

$$\langle f_{1n} * g, \phi\rangle = -\sum_{k=1}^{r+[n/2]} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}),$$

and so

$$\lim_{n\to\infty}\langle f_{1n} * g, \phi\rangle = -\sum_{k=1}^{\infty} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}).$$

Next, if $r \geq -[n/2]$, then

$$\langle f_{2n} * g, \phi\rangle = -\sum_{k=r+1+[n/2]}^{n+r-1} 2^{r-k}\tau\left(\frac{k-r+\alpha(k-r)^{r-k}}{n}\right)\phi'(r-\alpha k^{-k}-\alpha(k-r)^{r-k}).$$

Thus

$$|\langle f_{2n} * g, \phi \rangle| \le M \sum_{k=r+1+[n/2]}^{n+r-1} 2^{r-k},$$

where

$$M = \sup\{|\phi(x)| : r - 1/2 < x < r + 1/2\}$$

and so

$$\lim_{n\to\infty} \langle f_{2n} * g, \phi \rangle = 0.$$

If $r < -[n/2]$, it follows similarly that

$$\lim_{n\to\infty} \langle f_{2n} * g, \phi \rangle = 0.$$

It can be proved similarly that

$$\lim_{n\to\infty} \langle f_{3n} * g, \phi \rangle = 0$$

for all r.

We have therefore proved that $\lim_{n\to\infty} \langle f_n * g, \phi \rangle$ exists for all test functions ϕ with supports contained in the interval $(r - \frac{1}{2}, r + \frac{1}{2})$. It now follows that the limit exists for all ϕ in $\mathcal{D}$, proving that the convolution product $f * g$ exists by Definition 2.

Now put

$$\begin{aligned} f_n(x) &= \sum_{k=1}^{\infty} 2^{-k} \delta'(x - k - \alpha k^{-k}) \tau_n(x) \\ &= \sum_{k=1}^{n-1} 2^{-k} \delta'(x - k - \alpha k^{-k}) + 2^{-n} \tau_n(n + \alpha n^{-n}) \delta'(x - n - \alpha n^{-n}) \\ &\quad -2^{-n} n^n \tau_n'(n + \alpha n^{-n}) \delta(x - n - \alpha n^{-n}). \end{aligned}$$

The convolution product $f_n * g$ exists by Definition 1 and

$$\begin{aligned} f_n * g &= \sum_{k=1}^{\infty} \sum_{i=1}^{n-1} 2^i \delta'(x + k - i + \alpha k^{-k} - \alpha i^{-i}) \\ &\quad + 2^{-n} \tau(\alpha) \sum_{k=1}^{\infty} \delta'(x + k - n + \alpha k^{-k} - \alpha n^{-n}) \\ &\quad - 2^{-n} n^n \tau'(\alpha) \sum_{k=1}^{\infty} \delta(x + k - n + \alpha k^{-k} - \alpha n^{-n}). \end{aligned}$$

Let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $(-1/2, 1/2)$

and $\phi(0) \neq 0$. Then

$$2^{-n}n^{n}\tau'(\alpha)\langle\sum_{k=1}^{\infty}\delta(x+k-n+\alpha k^{-k}-\alpha n^{-n}),\phi\rangle = 2^{-n}n^{n}\tau'(\alpha)\phi(0),$$

which is not a negligible function, unless $\tau'(\alpha)=0$ or $\phi(0)=0$. It follows that

$$\mathrm{N}-\lim_{n\to\infty}\langle f_n * g, \phi\rangle$$

does not exist for all ϕ in $\mathcal{D}$ and all τ. The convolution product $f * g$ therefore does not exist by Definition 3.

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REZIME

O DVE DEFINICIJE NEKOMUTATIVNOG KONVOLUTIVNOG PROIZVODA DISTRIBUCIJA

Data su dva uopštenja standardne definicije konvolutivnog proizvoda dve uopštene funkcije. Pokazano je da nijedno nije uopštenje drugog.

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THE DIAGONAL RELATION IN THE LATTICE OF WEAK CONGRUENCES AND THE REPRESENTATION OF LATTICES

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ABSTRACT.

The problem of representation of an algebraic lattice L by the lattice $C_w(A)$ of weak congruences of an algebra A (i.e. of all congruences on all the subalgebras of A) is closely related to the localization of a diagonal relation Δ in $C_w(A)$. Δ is a co-distributive element in $C_w(A)$ ([6]), and it can also be neutral ([6]) and exceptional (as shown here).

Here we shall discuss the following question:

If a is a co-distributive (neutral, exceptional) element of an algebraic lattice L, is there an algebra A, such that $C_w(A) \simeq L$, and that $f(a)=\Delta$ under that isomorphism?

* * *

AMS Sub Class. 06B15
Key words and phrases: reprezentation of lattices, special elements in a lattice

Every positive answer to this question gives a representation of L by $C_w(\mathcal{A})$.

We shall show that the answer is positive if a is a minimal element of L. Thus, every algebraic lattice has at least one respresentation by weak congruences. This representation turns out to be the only one for lattices having no co-distributive elements different from 1 and 0.

However, the general answer to the above mentioned question is negative, because of some algebraic properties of $C_w(\mathcal{A})$ that we prove here. In order to give a lattice-theoretic interpretation of these algebraic results, we define Δ-suitable element of an algebraic lattice as the one that satisfies all the mentioned necessary conditions under which an element corresponds to Δ.

1. INTRODUCTION.

A *weak congruence* ρ on an algebra $\mathcal{A}=(A,F)$ is a symmetric, transitive and compatible relation on $\mathcal{A}$, which is also weakly reflexive: if c is a constant on $\mathcal{A}$, then $c\rho c$ ([4]). We denote by $C_w(\mathcal{A})$, $C(\mathcal{A})$ and $S(\mathcal{A})$ the algebraic lattices of weak congruences, of (ordinary) congruences, and of subalgebras of $\mathcal{A}$, respectively.

$C_w(\mathcal{A})$ coincides with the lattice of all the congruences on all the subalgebras of $\mathcal{A}$(under set inclusion). Moreover, $C(\mathcal{A})$ is its sublattice (as a filter generated by $\Delta=\{(x,x)\mid x\in A\}$), and $S(\mathcal{A})$ is a retract of $C_w(\mathcal{A})$ (ideal generated by Δ)([4]). Subalgebras are represented in $C_w(\mathcal{A})$ by diagonal relations ($\mathcal{B}\in S(\mathcal{A})$ which correspond to $d_{B^2}=\{(x,x)\mid x\in B\}$).

We shall use the following special elements of a bounded lattice L: if $a\in L$ then

(I) a is *co-distributive*, if for all $x,y\in L$

$$a\wedge(x\vee y)=(a\wedge x)\vee(a\wedge y),$$

or, equivalently, if $m_a: x\to x\wedge a$ is a homomorphism ([1]);

thus, a co-distributive element uniquely determines the following sublattices of L: (a], [a) (ideal and filter generated by a), and for every x∈L $[x]_{\theta_a}$, where θ_a is a congruence on L induced by m_a ;

(II) a is *neutral*, if $m_a: x \to x \wedge a$, and $n_a: x \to x \vee a$ are homomorphisms, and $f_a: x \to (x \wedge a, x \vee a)$ is an embedding of L into (a]x[a) ([1]);

(III) a is *exceptional*, if it is neutral and the classes of the congruence induced by m_a have maximum elements which form a sublattice M_a of L ([3]).

For x∈L, let $\overline{x}$ be the maximum of the class to which x belongs; thus $m_a(x)=m_a(\overline{x})$.

Considering the algebraic lattice $C_w(A)$ of an algebra A, we get the following statements, the proofs of which are straightforward.

1. Δ is a co-distributive element of $C_w(A)$.

2. The mapping $m_\Delta: \rho \to \rho \wedge \Delta (\rho \in C_w(A))$ is a lattice homomorphisam (the homomorphic image is $S(A)$, as already pointed out).

3. The mapping $n_\Delta: \rho \to \rho \vee \Delta (\rho \in C_w(A))$ is a lattice homomorphism if and only if A satisfies the Congruence Intersection Property (CIP) ([4]). In that case, Δ is a distributive elements of $C_w(A)$.

4. Δ is neutral in $C_w(A)$ if and only if A satisfies both CIP and CEP (the Congruence Extension Property) ([6]).

5. The classes of the congruence induced by m_Δ have maximum elements forming a set $M_\Delta=\{B^2 \mid B\in S(\mathcal{A})\}$ (M_Δ is not necessarily a sublattice of $C_w(\mathcal{A})$).

***

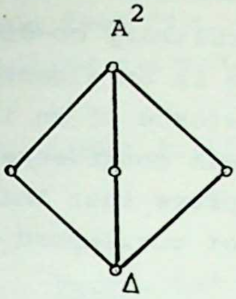
Now, let L be an arbitrary algebraic lattice. In order to give an answer to the question at the beginning of the article, we shall start with the fact that the minimal element, 0, is co-distributive in L. Thus, we have the following Represetation Theorem.

Theorem 1. *Let* L *be an algebraic lattice. Then there is an algebra* $\mathcal{A}$ *such that its lattice of weak congruence* $C_w(\mathcal{A})$ *is isomorphic to* L.

Proof. If L is algebraic, then there is an algebra $\mathcal{A}_1=(A,F_1)$ such that its lattice of (ordinary) congruences $C(\mathcal{A}_1)$ is isomorphic to L (G.Grätzer, E.T.Schmidt, [2]. Now, let $\mathcal{A}=(A,F)$ be the algebra constructed on the same set A as $\mathcal{A}_1$, and with the same set of non-nullary operations, but in which the set of constants K is the whole set A (K=A), i.e. such that $F=A\cup F_1$. Then, obviously, $C_w(\mathcal{A})=C(\mathcal{A})=C(\mathcal{A}_1)$, and we are done. □

If L has no co-distributive element other than 0 and 1, then the only way to respereesent it by weak congruences is the one given in Theorem 1.

EXAMPLE 1. The lattice M_3(Fig.1) can be represented by the latice of weak congruences only by adding to the set of operations all the elements of a suitable algebra (Klein´s group, for example) as constants.



$A=(\{e,a,b,ab\},\cdot,^{-1},e,a,b,ab)$

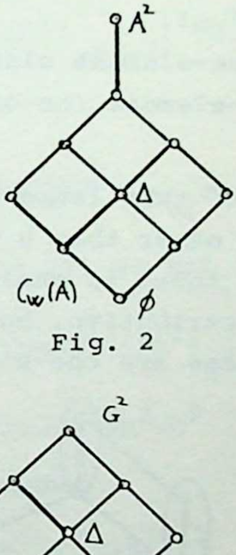
	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

$C_w(A)=C(A)$

Fig. 1

In the following two examples Δ is a neutral and exceptional element (respectively) in the lattice of weak congruences of the corresponding algebra. An example in which Δ is co-distributive (but not neutral) is $C_w(S_3)$, where S_3 is a symmetric group on a three-element set (see [5]).

Δ-neutral,
but not
exceptional

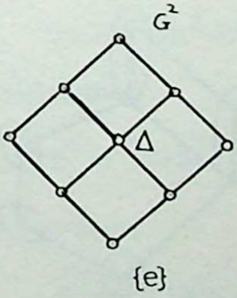


$A=(A,f_1,f_2)$ $A=\{a,b,c,d\}$

A	a	b	c	d
f_1	a	a	d	c

A	a	b	c	d
f_2	b	a	d	c

Fig. 2



G - any cyclic group of order p·q p,q-prim, p≠q.

Δ- exceptional

C_wG.

Fig. 3

However, the localization of an arbitrary co-distributive (neutral, exceptional) element a in an algebraic lattice L is not sufficient for the existence of an isomorphism f between L and the lattice of weak congruences of an algebra (with $f(a)=\Delta$). In order to prove that there are co-distributive elements which could not correspond to Δ, we advance the following propositions.

LEMMA 2. *Let* $\rho_1,\rho_2\in C_w(A), \rho_1\neq\emptyset$ 1) *for an arbitrary algebra* A, *and let* θ_Δ *be the congruence on* $C_w(A)$ *induced by* m_Δ *(see 2.). Then,* $[\rho_1]_{\theta_\Delta}\lneq[\rho_2]_{\theta_\Delta}$ *implies that* $|[\rho_2]_{\theta_\Delta}|\neq 1$.

(In other words, there are no pairs of one-element comparable classes modulo θ_Δ, unless the first contains the empty relation.)

Proof. Obvious, since a one-element class consists of a diagonal relation of a one-element (or empty) subalgebra. □

An immediate consequence of this lemma is that in a bounded chain the only element other than 0 which could correspond to Δ is the atom (if there is one). All the other elements in a chain are co-distributive, but the correspoding classes of weak congruences are one-element and

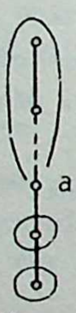


Fig. 4

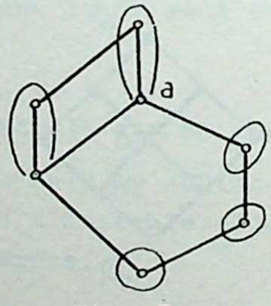


Fig. 5

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1) It is possible that the weak congruence is an empty relation (see[4]).

comparable (see Fig. 4 and 5 for the lattices in which co-distributive elements violate Lemma 2).

LEMMA 3. *If* $B,C\in S(A)$ *for an algebra* A, *then* $B\wedge C\neq\emptyset$ *implies* $B^2\vee C^2=(B\vee C)^2$ *in* $C_w(A)$.

Proof. Let $B\wedge C=D\neq\emptyset$. Then

$B^2\vee C^2,(B\vee C)^2\in C(B\vee C)$ (the lattice of ordinary congruences on $B\vee C$). Now, from

$$[D^2]_{B^2\vee C^2}\in S(B\vee C)$$

($[D^2]_{B^2\vee C^2}$ is a class in the congruence $B^2\vee C^2$ to which D^2 belongs), it follows that

$$([D^2]_{B^2\vee C^2})^2\leqslant B^2\vee C^2 \ . \tag{1}$$

Clearly,

$$B,C\leqslant [D^2]_{B^2\vee C^2} \ . \tag{2}$$

From (1) and (2) we get

$$(B\vee C)^2\leqslant([D^2]_{B^2\vee C^2})^2\leqslant B^2\vee C^2 \ ,$$

i.e. $(B\vee C)^2\leqslant B^2\vee C^2$.

Since obviously

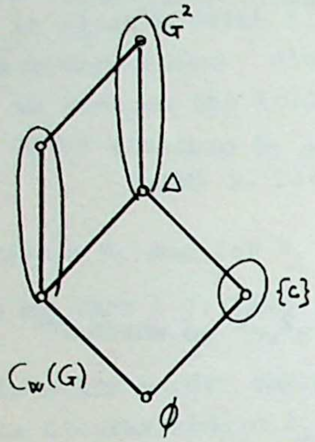
$$B^2\vee C^2\leqslant(B\vee C)^2 \ ,$$

finally we have

$$B^2\vee C^2=(B\vee C)^2 \ . \ \square$$

COROLLARY 4. *If* $\cap(B\,|\,B\in S(A))\neq\emptyset$, *then* M_Δ *is a sublattice of* $C_w(A)$. □

REMARK. The converse of Corallary 4 is not valid: see the groupoid on Fig. 6. M_Δ is there a sublattice of $C_w(G)$, but the minimal element of this lattice is the empty set.



$(G,\cdot)$ $G=\{a,b,c\}$

	a	b	c
a	b	b	c
b	b	a	a
c	a	a	c

Fig. 6

COROLLARY 5. *If Δ is neutral element in $C_w(A)$, and the minimal subalgebra of A is not empty, then Δ is exceptional in $C_w(A)$.*

Proof. By Corollary 4 and by (III). □

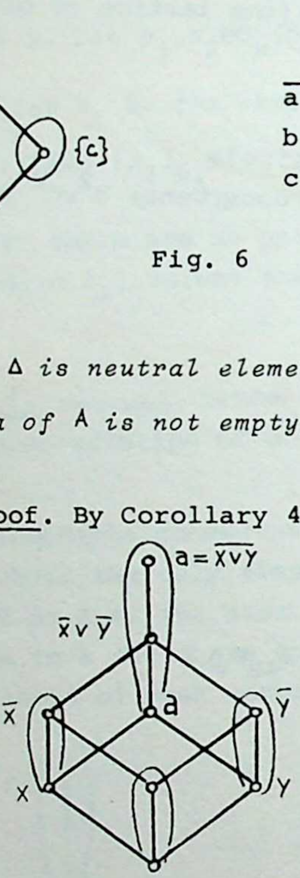


Fig. 7

Lemma 3, and Corollaries 4 and 5 exclude a wide class of co-distributive elements as possible diagonal relations in the representation of lattices by weak congruences. A co-distributive element a in the lattice in Fig. 7 violates Corollary 5, since it is neutral, the minimal class of the congruence induced by m_a has two elements (the corresponding minimal subalgebra could not be void), but a is not exceptional.

For an arbitrary algebra $\mathcal{A}$, and $\mathcal{B}\in S(\mathcal{A})$, $\theta\in C(\mathcal{B})$, let[1]

$$B[\theta] \overset{def}{=} \{ a\in A \mid (\exists b\in B)\, a\theta b\}.$$

LEMMA 6. a) $\mathcal{B}\leqslant B[\theta]\leqslant \mathcal{A}$;

b) *if* $\{\theta_i, i\in I\}\leqslant C(\mathcal{A})$, *and* $B[\theta_i]=B$ *for every* i, *then* $B[\bigvee_{i\in I}\theta_i]=B$.

Proof.

a) Obvious.

b) If $a\in B[\vee\theta_i]$, then there are $a=a_1,\dots,a_{n-1}\in A$, $a_n=b\in B$, and $\theta_{i_1},\dots,\theta_{i_{n-1}}\in\{\theta_i, i\in I\}$, such that $a_j\theta_{i_j}a_{j+1}$. If $a\notin B$, then $a_j\notin B$ for some $j\in\{1,\dots,n-1\}$. But this contradicts the assumption $B[\theta_{i_j}]=B$, since $a_j\theta_{i_j}a_{j+1}$. □

For $\mathcal{B}\in C(A)$, let

$$\theta_B \overset{def}{=} \bigvee(\theta\in C(A) \mid B[\theta]=B).$$

Obviously, $\theta_B\in C(A)$, and for every $\theta\in C(A)$,

a) $B[\theta]=B \leftrightarrow \theta\leqslant\theta_B$.

Moreover,

b) $B[\theta_B]=B$.

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[1] The consideration that follows, concluding with Theorem 9, was suggested by E. Kiss.

(θ_B is thus the greatest congruence on A in which the subalgebra B is a union of some classes.)

LEMMA 7. *For every* $B\in S(A), \theta\in C(A)$

$$B^2\vee\theta=B[\theta]^2\vee\theta, \textit{ in } C_w(A)$$

Proof. By the definition of $B[\theta]$,

$$B[\theta]^2\leq\bigcap(\rho\in C(A)\mid B^2\vee\theta\leq\rho)=B^2\vee\theta.$$

Thereby, $B[\theta]^2\vee\theta\leq B^2\vee\theta$.

The opposite inequality is trivial. □

COROLLARY 8. *If* $B[\theta]=A$, *then* $B^2\vee\theta=A^2$. □

THEOREM 9. *For an algebra* A, *let* $B\in S(A)$, *and* $A\succ B$ (B *is a coatom in* $S(A)$ *Then,*

$$\vee(\theta\in C(A)\mid\theta\vee B^2<A^2)\neq A^2.$$

Proof. For $B\in S(A)$, $A\succ B$, and $\theta\in C(A)$ $\theta\vee B^2<A^2$ implies $B[\theta]\neq A$ (by Corollary 8). Hence, $B[\theta]=B$ (since, by assumption, there is no subalgebra between A and B). Thus, $\theta_B\geq\theta$, and $A^2>\theta_B\geq\vee(\theta\in C(A)\mid\theta\vee B^2<A^2)$. □

So far there is no name for an element which satisfies all the listed necessary conditions under which it corresponds to a diagonal relation in a representation of a lattice by weak congruences. Therefore, we shall give the following definition (for the notation, see the introduction).

DEFINITION 1. An element a of an algebraic lattice L is Δ-*suitable* if it is co-distributive and satisfies the following conditions:

i) $x \wedge y \neq 0$ implies $\overline{x} \vee \overline{y} = \overline{x \vee y}$, for all $x, y \in L$;

ii) if $x \neq 0$ and $\overline{x} < y$, then $\overline{y \wedge a} \neq y \wedge a$, for all $x, y \in L$;

iii) if $x \neq 0$ and $x = \wedge(y \mid y \in L \setminus \{0\})$, then $x \neq \overline{x}$, for every $x \in L$;

iv) if $x \in L$ and $\overline{x} \prec 1$ ($\overline{x}$ is covered by 1), then $\vee(y \in L \mid y \geq a, y \vee \overline{x} \prec 1) \neq 1$.

It is clear that a Δ-suitable element must be co-distributive. Condition i) is required by Lemma 3, and ii) as well as iii) by Lemma 2; iv) follows by Theorem 9.

Obviously, a Δ-suitable element which is neutral, must be exceptional, if $0 \neq \overline{0}$.

As a final example, we note that in a free distributive lattice with three generators ($F_D(3)$, see Fig.8), these generators are the only Δ-suitable elements. Since a Δ-suitable element satisfies only some necessary conditions under which it corresponds to a diagonal relation, it is still an open problem whether these conditions are sufficient, as well. To be precise:

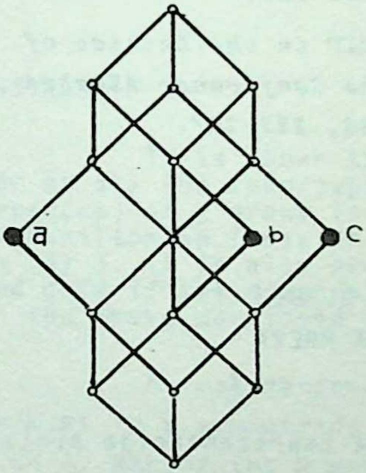

Fig. 8

PROBLEM. If a is a Δ-suitable element of an algebraic lattice L, is there an algebra A, such that $C_w(A) \simeq L$, and that $f(a) = \Delta$ under that isomorphism?

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REZIME

DIJAGONALNA RELACIJA U MREŽI SLABIH KONGRUENCIJA I REPREZENTACIJA MREŽA

U radu je dokazana teorema reprezentacije proizvoljne algebarske mreže uz pomoć mreže slabih kongruencija algebre. S obzirom da to predstavljanje zavisi od lokalizovanja elementa mreže koji se u reprezentaciji preslikava u dijagonalnu relaciju, daju se neki potrebni uslovi pod kojima proizvoljan kodistributivan element algebarske mreže odgovara dijagonali. Definiše se odgovarajići Δ-podoban element proizvoljne algebarske mreže.

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A NOTE ON SOME LATTICE CHARACTERIZATIONS OF HAMILTONIAN GROUPS

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ABSTRACT

It is shown that the lattice of all the congruences on all the subgroups (i.e. the lattice of all weak congruences) of a group is modular if and only if the group is Hamiltonian (this is the solution of a problem stated in [4]). It is also proved that a group is Hamiltonian if and only if its diagonal relation is an exceptional element in the above-mentioned lattice.

A *weak congruence relation* ([3]) on an algebra $\mathcal{A}=(A,F)$ is a symmetric, transitive and compatible relation ρ on A, satisfying a weak reflexivity: if c is a constant in $\mathcal{A}$, then

$c\rho c$.

For a given algebra $\mathcal{A}$, let $S(\mathcal{A})$, $C(\mathcal{A})$, and $C_w(\mathcal{A})$

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be the lattices of subalgebras, congruences and weak congruences on $\mathcal{A}$, respectively.

It is clear that $C_w(\mathcal{A})$ is an algebraic lattice under set inclusion and that it coincides with the lattice of all the congruences on all the subalgebras of $\mathcal{A}$.

$C(\mathcal{A})$ is a sublattice of $C_w(\mathcal{A})$, namely, it is a filter $[\Delta)$, where $\Delta=\{(x,x) \mid x\in A\}$ (diagonal relation on $\mathcal{A}$).

$S(\mathcal{A})$ is isomorphic with the ideal $(\Delta]$, under the mapping $B\to d_{B^2}$, where $d_{B^2}=\{(x,x) \mid x\in B\}$ (a diagonal relation on $\mathcal{B}\in S(\mathcal{A})$).

Moreover, $S(\mathcal{A})$ is a homomorphic image of $C_w(\mathcal{A})$, under $\rho\to\mathcal{B}$, where $d_\rho=d_{B^2}$, and $d\rho \overset{\mathrm{def}}{=} \rho\wedge\Delta$ (a diagonal of ρ).

In the following, we shall identify the subalgebras of $\mathcal{A}$ with the corresponding diagonal relations in $C_w(\mathcal{A})$.

$\mathcal{A}$ is said to have the *congruence intersection property* (CIP) ([3]) if for all $\rho,\theta\in C_w(\mathcal{A})$

$$(\rho\wedge\theta)_A=\rho_A\wedge\theta_A,$$

where $\rho_A \overset{\mathrm{def}}{=} \cap(\sigma\in C(\mathcal{A}) \mid \rho\subset\sigma)$.

Obviously, $\rho_A=\rho\vee\Delta$, and thus CIP expresses a distributivity property of Δ.

Recall that $\mathcal{A}$ satisfies the *congruence extension property* (CEP) if every congruence on an arbitrary subalgebra of $\mathcal{A}$ is a restriction of some congruence on $\mathcal{A}$.

It was proved in [3] that:

(I) An algebra $\mathcal{A}$ has a modular lattice of weak congruences if and only if $\mathcal{A}$ satisfies CEP and CIP, and both $S(\mathcal{A})$ and $C(\mathcal{A})$ are modular lattices.

It was proved in [5] that:

(II) $\mathcal{A}$ satisfies CEP and CIP if and only if the mapping $\rho\to(d\rho,\rho_A)$ is an embedding from $C_w(\mathcal{A})$ into $S(\mathcal{A})\times C(\mathcal{A})$.

If G is a group, then every $\rho \in C_w(G)$ uniquely determines a pair (H,K) of subgroups of G, where $K \triangleleft H$, $K=[e]_\rho$ (an equivalence class containing a neutral element e - constant in G), and H is represented by a diagonal relation d_ρ, as mentioned before. Conversely, every pair (H,K) of subgroups of G, such that K is normal in H, determines one weak congruence of G, namely the congruence on H which corresponds to $K \triangleleft H$. Thus, the lattice $C_w(G)$ is isomorphic with the lattice of all pairs (H,K), where H runs over all the subgroups of G, and K over the normal subgroups of H.

Now let $\overline{K}$ be a least normal subgroup of G containing a subgroup K i.e. let

$$\overline{K} \overset{def}{=} \cap (K_i \triangleleft G \mid K \leq K_i).$$

PROPOSTITION 1. *A group* G *satisfies* CEP *and* CIP *if and only if* $K \to \overline{K}$ *is an embedding from S(G) into* C(G) (the latter considered as a lattice of all the normal subgroups of G).

Proof.

By the above-mentioned isomorphism between $C_w(G)$ and the lattice of ordered pairs of subgroups, the embedding from (II) can be given by $(H,\overline{K}) \to (H,\overline{K})$, $K \triangleleft H \leq G$, and hence by $K \to \overline{K}$. □

COROLLARY 2. *A group* G *satisfies* CEP *and* CIP *if and only if it is Hamiltonian.*

Proof.

If G satisfies CEP and CIP, then the embedding $K \to \overline{K}$ from S(G) into C(G) maps the normal subgroup into itself. Thus, S(G)=C(G), and G is Hamiltonian. The converse follows from Proposition 1. □

Corollary 2., together with (I), yields the following.

THEOREM 3. *A group is Hamiltonian if and only if its lattice of weak congruences is modular.*□

PROBLEM . Characterize groups satisfying CIP, which in this case has the following form for any two subgroups H,K of G,

$$\overline{H\cap K} = \overline{H}\cap\overline{K} .$$

(If G is finite, then CIP is equivalent with being Hamiltonian, see [4]).

Some other characterization of Hamiltonian groups can be given by means of the neutral and exceptional elements of the lattice $C_w(G)$.

Recall that an element a of a bounded lattice L (with 0 and 1) is said to be *neutral* if it satisfies the identity

$$(a\wedge x)\vee(x\wedge y)\vee(y\wedge a)=(a\vee x)\wedge(x\vee y)\wedge(y\vee a),$$

for all $x,y\in L$.

One can prove that:

a) a is neutral in L if and only if the mappings

$$m_a:x\to x\wedge a \quad\text{and}\quad n_a:x\to x\vee a \quad (x\in L)$$

are homomorphisms and

$$f_a:x\to(x\wedge a,x\vee a)$$

is an embedding from L into $(a]\times[a)$.

An element a of a bounded lattice L is *exceptional* if it is neutral and the classes of the congruence induced by m_a have maximum elements which form a sublattice of L(see[2]).

Now, if we consider the bounded lattice $C_w(A)$ of an algebra A, and the diagonal relation $\Delta\in C_w(A)$, then we have the following statements, the proofs of which are straightforward:

a) $m_\Delta : \rho \to \rho \wedge \Delta$ $(\rho \in C_w(A))$ is a lattice homomorphism (a homomorphic image is $S(A)$, as already stated).

2) $n_\Delta : \rho \to \rho \vee \Delta$ $(\rho \in C_w(A))$ is a lattice homomorphism if and only if A satisfies CIP.

3) Δ is a neutral element of the lattice $C_w(A)$ if and only if A satisfies CEP and CIP (see[4]).

4) The classes induced by the congruence m_Δ in $C_w(A)$ have maximal elements; the set of such elements is

$$M_\Delta = \{B^2 \mid B \leqslant A\}.$$

(M_Δ is not necessarily a sublattice of $C_w(A)$).

In the case of groups, this yields the following:

LEMMA 4. *If* G *is a group, then* $M_\Delta = \{H^2 \mid H \leqslant G\}$ *is a sublattice of* $C_w(G)$.

Proof.

Since $H^2 \cap K^2 = (H \cap K)^2$, for the subgroups H,K of G, all we have to prove is that in the lattice $C_w(G)$

$$(H \vee K)^2 \leqslant H^2 \vee K^2$$

(since obviously $H^2 \vee K^2 \leqslant (H \vee K)^2$).

If e is a neutral element of G, then

$$(H \vee K)^2 = ([e]_{(H \vee K)^2})^2 \leqslant ([e]_{H^2 \vee K^2})^2 = H^2 \vee K^2 \ . \ \square$$

THEOREM 5. *A group* G *is Hamiltonian if and only if* Δ *is a neutral element of the lattice* $C_w(G)$.

Proof.

By 3) and Corollary 2. □

COROLLARY 6. *A group* G *is Hamilionian if and only if* Δ *is an exceptional element of the lattice* $C_w(G)$.

Proof.

By Lemma 4 and Theorem 5. □

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REZIME

O NEKIM MREŽNIM KARAKTERIZACIJAMA HAMILTONOVIH GRUPA

U radu je dokazano da je proizvoljna grupa Hamiltonova ako i samo ako je mreža njenih slabih kongruencija modularna.

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n–КВАЗИГРУППЫ БОЛА

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РЕЗЮМЕ

В работе рассматриваются n-квазигруппы (Q,A), $n \in N\setminus\{1\}$, удовлетворяющие тождествами

$$A(A(\overset{n-1}{x},A(y,\overset{n-1}{x})),z_1^{n-1}) = A(\overset{n-1}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-1})))$$

и

$$A(z_1^{n-1},A(A(\overset{n-1}{x},y),\overset{n-1}{x})) = A(A(A(z_1^{n-1},x),\overset{n-2}{x},y),\overset{n-1}{x}),$$

которые для $n = 2$ превращаются в левое и правое тождество Бола [2-4]. Автор позволил себе (Q,A) назвать n-квазигруппой Бола. Доказано, что (Q,A), $n \in N\setminus\{1\}$, обладает $(n-2)$-арной опперацией $e_{\{1,n\}}$ удовлетворяющей условию:

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(e_{\{1,n\}}(a_1^{n-2}),a_1^{n-2},x) = x \wedge$$
$$\wedge\ (A(x,a_1^{n-2},e_{\{1,n\}}(a_1^{n-2})) = x);$$

$e_{\{1,n\}}$ называется $\{1,n\}$-нейтральная операция n-группоида (Q,A), введена автором в [1]. (Таким образом, n-квазигруппы Бола для $n = 2$ являются лупами Муфанг.) Притом, $e_{\{1,n\}}$ для $n = 3$ подстановка множества Q, и $(Q,e_{\{1,n\}})$ $(n-2)$-квазигруппа для $n > 3$. В основе построения n-квазигрупп Бола для $n \in N\setminus\{1,2\}$: теорема 9 и теорема 10.

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Key words and phrases: quasigroups, groups, Moufang loops, $\{i,j\}$-neutral operations on n-groupoids.

*

ОПРЕДЕЛЕНИЕ 1. [1] Пусть (Q,A) n-группоид, $n \in N\setminus\{1\}$. Пусть, далее, $e_{\{1,n\}}$ отображение множества Q^{n-2} в множество Q. $(n-2)$-арная операция $e_{\{1,n\}}$ $\{1,n\}$-нейтральная операция n-группоида (Q,A) тогда и только тогда, когда имеет место формула:

$$(1n)\qquad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(e_{\{1,n\}}(a_1^{n-2}),a_1^{n-2},x)^{1)} = x \wedge$$

$$\wedge\ (A(x,a_1^{n-2},e_{\{1,n\}}(a_1^{n-2})) = x)^{2).3)}$$

Если $n = 2$, то $(1n)$ превращается в

$$(1_2)\qquad (\forall x \in Q)(A(e_{\{1,2\}}(\boxdot),x) = x \wedge A(x,e_{\{1,2\}}(\boxdot)) = x).$$

Таким образом, если $n = 2$, то речь идет об определении единицы группоида, т.е. об определении нульарной операции $e_{\{1,2\}}$ – взятие единицы $e_{\{1,2\}}(\boxdot)$.$^{4)}$

ЛЕММА 1_1. [1] В n-группоиде (Q, A), $n \in N\setminus\{1\}$, не существует больше чем одна $\{1,n\}$-нейтральная операция.$^{5)}$

ЛЕММА 1_2. [1] Пусть (Q,A) n-группоид, $n \in N\setminus\{1\}$. Пусть, далее, существуют отображения $e_{(1,n)}: Q^{n-2} \to Q$ и $e_{(n,1)}: Q^{n-2} \to Q$ удовлетворяющие условиям:

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1) Последовательность $g_p,\ldots,g_q$ (где $p,\ldots,q$ последовательность натуральных чисел следующих друг за другом) для краткости обозначаем через g_p^q; в частности: $g_p^p = g_p$. Подобно, последовательность $a,\underset{p}{\ldots},a$ обозначаем через $\overset{p}{a}$. Далее, под g_p^{p-1}, $p \in N$, и под $\overset{\circ}{g}$ понимаем пустую последовательность (отсуствие последовательности). Притом, если речь идет о, например, g_1^3, e_1^0 (или $g_1^3, \overset{\circ}{e}$), то g_1^3, e_1^0 (или $g_1^3, \overset{\circ}{e}$) считаем последовательностью g_1^3, и если речь идет о e_1^0 (или $\overset{\circ}{e}$) в некоторых случаях будем обозначать через $\boxdot$; например $F(a_1^0)$ будем обозначать через $F(\boxdot)$.

2) $$\Leftrightarrow (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(e_{\{1,n\}}(a_1^{n-2}),a_1^{n-2},x) = x) \wedge$$

$$\wedge\ (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(x,a_1^{n-2},e_{\{1,n\}}(a_1^{n-2})) = x).$$

3) $\{1,n\}$-нейтральная операция n-группоида является частным случаем $\{i,j\}$-нейтральной операцией n-группоида, введеной автором в [1].

4) т.е. о определению отображения $e_{\{1,2\}}: Q \to Q$, $Q^0 \overset{\text{деф}}{=\!=} \{\boxdot\}$, удовлетворяющего условию (1_2). Иначе, $e_{\{1,2\}}(\boxdot) \in Q$, обозначаем только через e.

(1_{Ln}) $(\forall a_i \in Q)_1^{n-2} (\forall x \in Q) A(e_{(1,n)}(a_1^{n-2}), a_1^{n-2}, x) = x$ и

(1_{Rn}) $(\forall a_i \in Q)_1^{n-2} (\forall x \in Q) A(x, a_1^{n-2}, e_{(n,1)}(a_1^{n-2})) = x$.

Тогда имеют место равенства:

$$e_{(1,n)}(a_1^{n-2}) = e_{(n,1)}(a_1^{n-2}) = e_{\{1,n\}}(a_1^{n-2})$$

для любого $(a_1^{n-2}) \in Q^{n-2}$, где $e_{\{1,n\}}$ $\{1,n\}$-нейтральная операция n-группоида (Q,A). [6]

В любом группоиде (2-группоиде) $e_{\{1,2\}}$ является нульарной операцией. Если в 3-квазигруппе (Q,A) существует $\{1,3\}$-нейтральная операция $e_{\{1,3\}}$, то $e_{\{1,3\}}$ является подстановкой множества Q. Именно, так как уравнение

$$A(e_{\{1,3\}}(x), x, a) = a$$

для любых $e_{\{1,3\}}(x) = b \in Q$ и $a \in Q$ обладает (единственным) реюением по неизвестной x, то $e_{\{1,3\}}$ отображение на. Далее, так как из предположения, что существуют $x_1 \in Q$ и $x_2 \in Q$ удовлетворяющие равенствам

$$e_{\{1,3\}}(x_1) = b \text{ и } e_{\{1,3\}}(x_2) = b,$$

т.е., ввиду (1n), что имеют место равенства

$$A(b, x_1, a) = a \text{ и } A(b, x_2, a) = a,$$

находим, что $e_{\{1,3\}}$ является и отображением 1-1. Если в n-квазигруппе (Q,A), $n > 3$, существует $\{1,n\}$-нейтральная операция $e_{\{1,n\}}$, то $(Q, e_{\{1,n\}})$ (n-2)-квазигруппа. Именно, так как уравнение

$$A(e_{\{1,n\}}(a_1^{i-1}, x_i, a_i^{n-3}), a_1^{i-1}, x_i, a_i^{n-3}, a) = a$$

---

[5] Речь идет об частном случае утверждения 1 из [1].

[6] Речь идет об частном случае утверждения 2 из [1].

для любых $e_{\{1,n\}}(a_1^{i-1}, x_i, a^{n-3}) = b$, $a_1^{i-1}, a_i^{n-3}, a \in Q$ обладает (единственным) решением по неизвестной x_i, то уравнение

$$e_{\{1,n\}}(a_1^{i-1}, x_i, a_i^{n-3}) = b, \ i \in \{1, \dots, n-2\},$$

для любых $a_1^{n-3}, b \in Q$ обладает по меньшей мере одним решением по неизвестной x_i. Далее, наконец, из предположения; что существуют $x_i \in Q$ и $\overline{x}_i \in Q$ удовлетворяющие равенствам

$$e_{\{1,n\}}(a_1^{i-1}, x_i, a_i^{n-3}) = b \text{ и } e_{\{1,n\}}(a_1^{i-1}, \overline{x}_i, a_i^{n-3}) = b,$$

т.е., ввиду (1n), что имеют место равенства

$$A(b, a_1^{i-1}, x_i, a_i^{n-3}, a) = a \text{ и } A(b, a^{i-1}, \overline{x}_i, a_i^{n-3}, a) = a,$$

находим, что $\overline{x}_i = x_i$. Таким образом, мы доказали, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 2. Пусть (Q, A) n-квазигруппа и $n \in N\backslash\{1,2\}$. Пусть, далее, (Q, A) обладает $\{1,n\}$-нейтральной операцией $e_{\{1,n\}}$. Тогда:

1° если $n = 3$, то $e_{\{1,n\}}$ подстановка множества Q; и
2° если $n > 3$, то $(Q, e_{\{1,n\}})$ $(n-2)$-квазигруппа.

* *

ОПРЕДЕЛЕНИЕ 2. Пусть (Q, A) n-квазигруппа, $n \in N\backslash\{1\}$; [5]. (Q, A) назовем n-квазигруппой Бола тогда и только тогда, когда имеют место формулы:

nB_L $$(\forall x \in Q)(\forall y \in Q)(\forall z_i \in Q)_1^{n-1}(A(A(\overset{n-1}{x}, A(y, \overset{n-1}{x})), z_1^{n-1}) = A(\overset{n-1}{x}, A(y, \overset{n-2}{x}, A(x, z_1^{n-1}))); \text{ и}$$

nB_R $$(\forall x \in Q)(\forall y \in Q)(\forall z_i \in Q)_1^{n-1} A(z_1^{n-1}, A(A(\overset{n-1}{x}, y), \overset{n-1}{x})) = A(A(A(z_1^{n-1}, x), \overset{n-2}{x}, y), \overset{n-1}{x}).$$

Если $n = 2$, то формулы под nB_L и nB_R превращаются в формулы:

$\underline{2B}_L$ $$(\forall x \in Q)(\forall y \in Q)(\forall z_1 \in Q)A(A(x,A(y,x)),z_1) = A(x,A(y,A(x,z_1)))$$ 7) ; и

$\underline{2B}_R$ $$(\forall x \in Q)(\forall y \in Q)(\forall z_1 \in Q)A(z_1,A(A(x,y),x)) = A(A(A(z_1,x),y),x)$$ 8).

ТЕОРЕМА 3. Если (Q,A) n-квазигруппа Бола, $n \in N\setminus\{1\}$, то (Q,A) обладает (в точности одной 9)) $\{1,n\}$-нейтральной операцией.

Доказательство.

Пусть (Q,A) n-квазигруппа Бола, $n \in N\setminus\{1\}$. Пусть, далее, a, a_1^{n-2} любые элементы множества Q. Тогда, так как (Q,A) n-квазигруппа, уравнение

(а) $$A(a,a_1^{n-2},x) = a$$

обладает единстенным решением по неизвестной x. Решение этого уравнения обозначим через

(б) $$e^{(a)}_{(n,1)}(a_1^{n-2}).$$

Притом, для каждого $a \in Q$ $e^{(a)}_{(n,1)}$ является отображением множества Q^{n-2} в множество Q.

Пусть, далее, a фиксированный элемент множества Q. Тогда, так как (Q,A) n-квазигруппа, для каждого $b \in Q$ существует в точности один $y \in Q$ удовлетворяющий уравнению:

(ц) $$b = A(\overset{n-1}{a},A(y,\overset{n-1}{a})).$$

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7) В (Q,A), $n = 2$, имеет место левое тождество Бола; [2-4].

8) В (Q,A), $n = 2$, имеет место правое тождество Бола; [2-4].

9) см. Лемму 1_1.

Учитывая (а), (б) и (ц), ввиду nB_L из определения 2, находим, что имеет место следующая цепь равенств:

$$A(b,a_1^{n-2},e^{(a)}_{(n,1)}(a_1^{n-2})) =$$

$$= A(A(\overset{n-1}{a},A(y,\overset{n-1}{a})),a_1^{n-2},e^{(a)}_{(n,1)}(a_1^{n-2})) =$$

$$= A(\overset{n-1}{a},A(y,\overset{n-2}{a},A(a,a_1^{n-2},e^{(a)}_{(n,1)}(a_1^{n-2}))))=$$

$$= A(\overset{n-1}{a},A(y,\overset{n-2}{a},a)) =$$

$$= A(\overset{n-1}{a},A(y,\overset{n-1}{a})) =$$

$$= b,$$

откуда получаем, что имеет место формула:

$$(\forall a_i \in Q)_1^{n-2}(\forall b \in Q)A(b,a_1^{n-2},e^{(a)}_{(n,1)}(a_1^{n-2})) = b$$

Притом, так как $e^{(a)}_{(n,1)}$ одно и то же отображение для каждого $a \in Q$, $e^{(a)}_{(n,1)}$ обозначим через $e_{(n,1)}$. В самом деле, мы доказали, что в n-квазигруппе Бола (Q,A) существует отображение $e_{(n,1)}$: $Q^{n-2} \to Q$ удовлетворяющее условию:

(д) $\quad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)A(x,a_1^{n-2},e_{(n,1)}(a_1^{n-2})) = x$ [10].

Подобним способом, учитывая nB_R, доказывается, что существует и отображение $e_{(1,n)}:Q^{n-2} \to Q$ удовлетворяющее условию:

(д') $\quad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)A(e_{(1,n)}(a_1^{n-2}),a_1^{n-2},x) = x$ [11].

Так как мы доказали, что существуют отображения $e_{(n,1)}$: $Q^{n-2} \to Q$ и $e_{(1,n)}:Q^{n-2} \to Q$ удовлетворяющие, в том же порядке, условиям (д) и (д'), ввиду лемы 1_2, существует отображение $e_{\{1,n\}} : Q^{n-2} \to Q$ такое, что

$\bar{I} \quad e_{\{1,n\}} = e_{(n,1)} = e_{(1,n)}$; и

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[10] см. формулу под (1_{Rn}) из леммы 1_2.

$\overline{2}$ $e_{\{1,n\}}$ является $\{1,n\}$-нейтральной операцией в n-квазигруппе Бола (Q,A). Притом, ввиду леммы 1_1, $\{1,n\}$-нейтральная операция $e_{\{1,n\}}$ однозначно определена.

Теорема доказана.

* * *

Пусть (Q,B) квазигруппа. Пусть, далее, k фиксириованный элемент множества Q. Тогда $L_B^{(k)}$ и $L_B^{(k)}$, определены следующим образом

$$(\alpha_1) \qquad v = L_B^{(k)} u \overset{\text{деф}}{=\!=} B(k,u) \text{ и}$$

$$(\alpha_2) \qquad v = R_B^{(k)} u \overset{\text{деф}}{=\!=} B(u,k),$$

подстановки множества Q; левая и правая трансляция квазигруппы (Q,B) [2-4]. Ввиду этого факта, учитывая (α_1) и (α_2), находим, что имеют место равенства:

$$(\bar{\alpha}_1) \qquad u = L_B^{(k)-1} v$$

$$(\bar{\alpha}_2) \qquad u = R_B^{(k)-1} v.$$

Так как (Q,B) квазигруппа, то (Q,B^{-1}) и $(Q,{}^{-1}B)$, где

$$(\beta_1) \qquad B^{-1}(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} B(x,z) = y \text{ и}$$

$$(\beta_2) \qquad {}^{-1}B(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} B(z,y) = x,$$

также квазигруппы; B^{-1} и ${}^{-1}B$, в том же порядке, правая и левая обратная операция операции B [2-5]. Отсюда, учитывая (α_1) и (α_2), находим, что имеют место равенства:

$$(\bar{\bar{\alpha}}_1) \qquad u = B^{-1}(k,v) \text{ и}$$

$$(\bar{\bar{\alpha}}_2) \qquad u = {}^{-1}B(v,k).$$

---

[11] см. формулу под (1_{Ln}) из леммы 1_2.

Наконец, учитывая $(\bar{\alpha}_1)$, $(\bar{\bar{\alpha}}_1)$, $(\bar{\alpha}_2)$ и $(\bar{\bar{\alpha}}_2)$, находим, что имеют место равенства:

$$L_B^{(k)-1}v = B^{-1}(k,v)$$

$$R_B^{(k)-1}v = {}^{-1}B(v,k)$$

для любого $v \in Q$.

Таким образом, имеет место:

ЛЕММА 4. Пусть (Q,B) квазигруппа. Пусть, далее, k фиксированний элемент множества Q. Тогда, если

$$L_B^{(k)}u \overset{\text{деф}}{=\!=} B(k,u) \text{ и } R_B^{(k)}u \overset{\text{деф}}{=\!=} B(u,k)$$

для каждого $u \in Q$, то

$$L_B^{(k)-1}u = B^{-1}(k,u) \text{ и } R_B^{(k)-1}u = {}^{-1}B(u,k)$$

для каждого $u \in Q$. Притом, B^{-1} и ${}^{-1}B$ определени через (β_1) и (β_2).

Учитывая определение 1, находим, что имеет место следующее утверждение:

ЛЕММА 5. Пусть (Q,A) n-квазигруппа обладающая $\{1,n\}$-нейтральной операцией $e_{\{1,n\}}$; $n \in N\setminus\{1,2\}$. Пусть, далее, c_1^{n-2} любые фиксированные элементы множества Q. Тогда $(Q,B_{(c_1^{n-2})})$, где

$$\text{(o)} \qquad B_{(c_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} A(x,c_1^{n-2},y)$$

для любых $x,y \in Q$, лупа[12] с единицей $e_{\{1,n\}}(c_1^{n-2})$.

ЛЕММА 6_1. Пусть (Q,A) n-квазигруппа Бола; $n \in N\setminus\{1,2\}$. Пусть, далее, a_1^{n-2} и b_1^{n-2} любые фиксированные элементы множества Q. Притом, пусть:

$$B_{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} A(x,a_1^{n-2},y) \text{ и}$$

---

[12] лупа - квазигруппа обладающая единицей [2-4].

$$B_{(b_1^{n-2})}(x,y) \overset{\text{деф}}{===} A(x,b_1^{n-2},y)$$

для любых $x,y \in Q$. Тогда имеют место равенства

$$(B_1) \qquad B_{(b_1^{n-2})}(x,y) = B_{(a_1^{n-2})}\left(x, L_{(a_1^{n-2})}^{(e_{\{1,n\}}(b_1^{n-2}))^{-1}} y\right) \text{ и}$$

$$(B_2) \qquad B_{(b_1^{n-2})}(x,y) = B_{(a_1^{n-2})}\left(R_{(a_1^{n-2})}^{(e_{\{1,n\}}(b_1^{n-2}))^{-1}} x, y\right)$$

для любых $x,y \in Q$, где $e_{\{1,n\}}$ нейтральная операция n-квазигруппы Бола (Q,A) [13].

Доказательство.

Сначала равенство под nB_L из определения 2 запишем в следующем виде:

$$(L) \qquad A(A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},x)),z_1^{n-2},z) =$$

$$= A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-2},z))).$$

Пусть a_1^{n-2}, b_1^{n-2}, c любые фиксированные элементы множества Q. Если в равенство под (L) положим

$$x = c \text{ и } z_1^{n-2} = a_1^{n-2},$$

учитывая определение под (o) из леммы 5, получаем равенство:

$$(a) \qquad B_{(a_1^{n-2})}(B_{(\overset{n-2}{c})}(c,B_{(\overset{n-2}{c})}(y,c)),z) =$$

$$= B_{(\overset{n-2}{c})}(c,B_{(\overset{n-2}{c})}(y,B_{(a_1^{n-2})}(c,z))).$$

Если, далее, в равенство под (L) положим

$$x = c \text{ и } z_1^{n-2} = b_1^{n-2},$$

учитывая определение под (o) из леммы 5, получаем равенство:

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[13] см. теорему 3.

(б) $$B_{(b_1^{n-2})}(B_{(\overset{n-2}{c})}(c,B_{(\overset{n-2}{c})}(y,c)),z) =$$

$$= B_{(\overset{n-2}{c})}(c,B_{(\overset{n-2}{c})}(y,B_{(b_1^{n-2})}(c,z))).$$

Ввиду леммы 5, определений трансляций квазигрупп ((α_1) и (α_2)) и факта, что трансляции квазигрупп подстановки носителей квазигрупп, равенства под (а) и (б) превращаются в равенства:

(а′) $$B_{(a_1^{n-2})}(y,L^{(c)-1}_{(a_1^{n-2})}\,z) = L^{(c)}_{(\overset{n-2}{c})}B_{(\overset{n-2}{c})}(R^{(c)-1}_{(\overset{n-2}{c})}L^{(c)-1}_{(\overset{n-2}{c})}y,z) \text{ и}$$

(б′) $$B_{(b_1^{n-2})}(y,L^{(c)-1}_{(b_1^{n-2})}\,z) = L^{(c)}_{(\overset{n-2}{c})}B_{(\overset{n-2}{c})}(R^{(c)-1}_{(\overset{n-2}{c})}L^{(c)-1}_{(\overset{n-2}{c})}\,y,z).$$

Из (а′) и (б′) следует равенство:

(ц) $$B_{(a_1^{n-2})}(y,L^{(c)-1}_{(a_1^{n-2})}\,z) = B_{(b_1^{n-2})}(y,L^{(c)-1}_{(b_1^{n-2})}\,z).$$

Если в (ц) положим $y = e_{\{1,n\}}(b_1^{n-2})$, ввиду (α_1) и леммы 5, находим, что имеет место равенство:

(д) $$L^{(e_{\{1,n\}}(b_1^{n-2}))}_{(a_1^{n-2})}\,L^{(c)-1}_{(a_1^{n-2})}\,z = L^{(c)-1}_{(b_1^{n-2})}\,z.$$

Из равенств (ц) и (д), учитывая факт, что трансляции квазигруппы (Q,B) подстановки множества Q, находим, что имеет место равенство

$$B_{(b_1^{n-2})}(x,y) = B_{(a_1^{n-1})}(x,L^{(e_{\{1,n\}}(b_1^{n-2}))^{-1}}_{(a_1^{n-2})}\,y).$$

Подобним способом, учитывая равенство под nB_R из определения 2, находим, что имеет место равенство

$$B_{(b_1^{n-2})}(x,y) = B_{(a_1^{n-2})}(R^{(e_{\{1,n\}}(b_1^{n-2}))^{-1}}_{(a_1^{n-2})}\,x,y).$$

Лемма доказана.

Ввиду леммы 6_1 и леммы 4, находим, что имеет место следующее утверждение:

ЛЕММА 6_2. Пусть (Q,A) n-квазигруппа Бола; $n \in N\backslash\{1,2\}$. Пусть, далее, a_1^{n-2} и b_1^{n-2} любые фиксированные элементы множества Q. Притом, пусть:

$$B_{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} A(x,a_1^{n-2},y) \quad \text{и}$$

$$B_{(b_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} A(x,b_1^{n-2},y)$$

для любых $x,y \in Q$. Тогда имеют место равенства:

$$(\bar{B}_1) \qquad B_{(b_1^{n-2})}(x,y) = B_{(a_1^{n-2})}(x,B^{-1}_{(a_1^{n-2})}(e_{\{1,n\}}(b_1^{n-2}),y)) \quad \text{и}$$

$$(\bar{B}_2) \qquad B_{(b_1^{n-2})}(x.y) = B_{(a_1^{n-2})}({}^{-1}B_{(a_1^{n-2})}(x,e_{\{1,n\}}(b_1^{n-2})),y)$$

для любых $x,y \in Q$, где $e_{\{1,n\}}$ нейтральная операция n-квазигруппы Бола $(Q,A)^{13)}$.

ЛЕММА 7. Пусть (Q,A) n-квазигруппа Бола, $n \in N\backslash\{1,2\}$. Пусть, далее, $e_{\{1,n\}}$ $\{1,n\}$-нейтральная операция n-квазигруппы $(Q,A)^{13)}$. Тогда, если

а) a_1^{n-2} любые фиксированные элементы множества Q; и

б) $B_{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} A(x,a_1^{n-2},y)$;

то:

1) $(Q,B_{(a_1^{n-2})})$ является лупой с единицей $e_{\{1,n\}}(a_1^{n-2})$; и

2) в $(Q,B_{(a_1^{n-2})})$ имеют место равенства

$$\text{GA1} \qquad B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(Fx,B_{(a_1^{n-2})}(y,x)),z) = B_{(a_1^{n-2})}(Fx,B_{(a_1^{n-2})}(y,B_{(a_1^{n-2})}(x,z)));$$

$$\text{GA2} \qquad B_{(a_1^{n-2})}(z,B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(x,y),\phi x)) = B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(z,x),y),\phi x); \quad \text{и}$$

$$\text{GA3} \qquad B_{(a_1^{n-2})}({}^{-1}B_{(a_1^{n-2})}(x,y),z) = B_{(a_1^{n-2})}(x,B^{-1}_{(a_1^{n-2})}(y,z))$$

для любых $x,y,z \in Q$, где $F : Q \to Q$ и $\Phi : Q \to Q$ определены следующим образом:

$$Fu \overset{\text{деф}}{=\!=} {}^{-1}B_{(a_1^{n-2})}(u, e_{\{1,n\}}(\overset{n-2}{u})) \quad \text{и}$$

$$\Phi u \overset{\text{деф}}{=\!=} B^{-1}_{(a_1^{n-2})}(e_{\{1,n\}}(\overset{n-2}{u}), u).$$

Доказательство.

Ввиду леммы 5, $(Q, B_{(a_1^{n-2})})$ лупа с единицей $e_{\{1,n\}}(a_1^{n-2})$.

Далее, равенства под nB_L и nB_R из определения 2 запишем в следующем виде:

$$\text{(L)} \qquad A(A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},x)),z_1^{n-2},z) =$$

$$= A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-2},z))); \text{ и}$$

$$\text{(R)} \qquad A(z,z_1^{n-2},A(A(x,\overset{n-2}{x},y),\overset{n-2}{x},x)) =$$

$$= A(A(A(z,z_1^{n-2},x),\overset{n-2}{x},y),\overset{n-2}{x},x),$$

где x,y,z,z_1^{n-2} любые элементы множества Q.

Учитывая определение

$$B_{(c_1^{n-2})}(u,v) \overset{\text{деф}}{=\!=} A(u,c_1^{n-2},v)$$

для любых $u,v,c_1^{n-2} \in Q$, равенства под (L) и (R) превращаются в равенства:

$$\text{(L')} \qquad B_{(z_1^{n-2})}(B_{(\overset{n-2}{x})}(x,B_{(\overset{n-2}{x})}(y,x)),z) =$$

$$= B_{(\overset{n-2}{x})}(x,B_{(\overset{n-2}{x})}(y,B_{(z_1^{n-2})}(x,z))); \text{ и}$$

$$\text{(R')} \qquad B_{(z_1^{n-2})}(z,B_{(\overset{n-2}{x})}(B_{(\overset{n-2}{x})}(x,y),x)) =$$

$$= B_{(\overset{n-2}{x})}(B_{(\overset{n-2}{x})}(B_{(z_1^{n-2})}(z,x),y),x)$$

для любых $x,y,z,z_1^{n-2} \in Q$.

Отсюда, учитывая условия а) и б), ввиду леммы 6_1, находим, что имеют место равенства:

(L")
$$B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(R_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}x, B_{(a_1^{n-2})}((R_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}y, x)), L_{(a_1^{n-2})}^{(e_{\{1,n\}}(z_1^{n-2}))^{-1}}z) =$$

$$= B_{(a_1^{n-2})}(R_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}x, B_{(a_1^{n-2})}((R_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}y, B_{(a_1^{n-2})}(x, L_{(a_1^{n-2})}^{(e_{\{1,n\}}(z_1^{n-2}))^{-1}}z))); \text{ и}$$

(R")
$$B_{(a_1^{n-2})}(R_{(a_1^{n-2})}^{(e_{\{1,n\}}(z_1^{n-2}))^{-1}}z, B_{(a_1^{n-2})}(B_{(a_1^{n-2})}((L_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}y), L_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}x)) =$$

$$= B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(R_{(a_1^{n-2})}^{(e_{\{1,n\}}(z_1^{n-2}))^{-1}}z, x), L_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}y, L_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}x)$$

для любых $x, y, z, z_1^{n-2} \in Q$.

Ввиду леммы 4, имеет место:

$$Y = L_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}y = B^{-1}_{(a_1^{n-2})}(e_{\{1,n\}}(\overset{n-2}{x}), y);$$

$$Y = R_{(a_1^{n-2})}^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}y = {}^{-1}B_{(a_1^{n-2})}(y, e_{\{1,n\}}(\overset{n-2}{x}));$$

$$Z = L_{(a_1^{n-2})}^{(e_{\{1,n\}}(z_1^{n-2}))^{-1}}z = B^{-1}_{(a_1^{n-2})}(e_{\{1,n\}}(z_1^{n-2}), z);$$

$$Z = R^{(e_{\{1,n\}}(z_1^{n-2}))^{-1}}_{(a_1^{n-2})} \qquad z = {}^{-1}B_{(a_1^{n-2})}(z, e_{\{1,n\}}(z_1^{n-2})).$$

Отсюда, так как $(Q, B_{(a_1^{n-2})})$ квазигруппа, находим, что

$$\alpha_x \overset{\text{деф}}{=} \{(y,Y) \mid Y = B^{-1}_{(a_1^{n-2})}(e_{\{1,n\}}(\overset{n-2}{x}), y) \wedge y \in Q\};$$

$$\beta_x \overset{\text{деф}}{=} \{(y,Y) \mid Y = {}^{-1}B_{(a_1^{n-2})}(y, e_{\{1,n\}}(\overset{n-2}{x}) \wedge y \in Q\};$$

$$\gamma_{z_1^{n-2}} \overset{\text{деф}}{=} \{(z,Z) \mid Z = B^{-1}_{(a_1^{n-2})}(e_{\{1,n\}}(z_1^{n-2}), z) \wedge z \in Q\};$$

и

$$\delta_{z_1^{n-2}} \overset{\text{деф}}{=} \{(z,Z) \mid Z = {}^{-1}B_{(a_1^{n-2})}(z, e_{\{1,n\}}(z_1^{n-2}) \wedge z \in Q\}$$

являются подстановками множества Q для каждого $x \in Q$ и $(z_1^{n-2}) \in Q^{n-2}$; $n \in N\backslash\{1,2\}$. Учитывая этот факт, находим, что равенства под (L") и (R") превращаются в равенства

(L"') $$B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(R^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}_{(a_1^{n-2})}x, B_{(a_1^{n-2})}(Y,x)), Z) =$$

$$= B_{(a_1^{n-2})}(R^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}_{(a_1^{n-2})}x, B_{(a_1^{n-2})}(Y, B_{(a_1^{n-2})}(x,Z))); \text{ и}$$

(R"') $$B_{(a_1^{n-2})}(Z, B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(x,Y), L^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}_{(a_1^{n-2})}x)) =$$

$$= B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(B_{(a_1^{n-2})}(Z,x), Y), L^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}_{(a_1^{n-2})}x)$$

для любых $x, Y, Z \in Q$. Отсюда, обозначая x, Y, Z,

$$R^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}_{(a_1^{n-2})}x \quad \text{и} \quad L^{(e_{\{1,n\}}(\overset{n-2}{x}))^{-1}}_{(a_1^{n-2})}x,$$

в томе же порядке, через x, y, z, Fx и Φx, притом, учитывая лемму 4, находим, что имеют место равенства под GA1 и GA2 для любых $x, y, z \in Q$, где

$$Fu = {}^{-1}B_{(a_1^{n-2})}(u, e_{\{1,n\}}(\overset{n-2}{u})) \quad \text{и}$$

$$\Phi u = B^{-1}_{(a_1^{n-2})}(e_{\{1,n\}}(\overset{n-2}{u}),u).$$

Наконец, учитывая лемму 6_1, лемму 6_2 и утверждение 2, находим, что имеет место и равенство под GA3 для $x,y,z \in Q$.

Лемма доказана.

ЛЕММА 8_1. Пусть (Q,B) лупа с единицей e, удовлетворяющая ревенствам

$\underline{GB}_L$ $\quad B(B(Fx,B(y,x)),z) = B(Fx,B(y,B(x,z)))$[14]; и

$\underline{GB}_R$ $\quad B(z,B(B(x,y),\Phi x)) = B(B(B(z,x),y),\Phi x)$[15]

для любых $x,y,z \in Q$, где F и Φ унарные операции в множестве Q. Тогда в (Q,B) имеют место равенства

$$B({}^{-1}x,B(x,z)) = z; \quad \text{и}$$

$$B(B(z,x),x^{-1}) = z$$

для любых $x,z \in Q$, где $B({}^{-1}x,x) = e$ и $B(x,x^{-1}) = e$, т.е. тогда (Q,B) IP-лупа [2-4].

Доказательство.

Если в равенство под GB_L положим $y = {}^{-1}x$, получаем равенство

$$B(Fx,z) = B(Fx,B({}^{-1}x,B(x,z)),$$

откуда, ввиду закона сокращения, находим, что имеет место равенство:

$$B({}^{-1}x,B(x,z)) = z.$$

---

[14] для $F = I$ равенство под GB_L превращается в левое тождество Бола; [2-4].

[15] для $\Phi = I$ равенство под GB_R превращается в правое тождество Бола; [2-4].

Подобно, если в равенство под GB_R положим $y = x^{-1}$, получаем равенство

$$B(z,\Phi x) = B(B(B(z,x),x^{-1}),\Phi x),$$

откуда, ввиду закона сокращения, находим, что имеет место равенство

$$B(B(z,x),x^{-1}) = z.$$

ЛЕММА 8_2. Пусть (Q,B) IP-лупа [2-4].[16] Тогда, если в (Q,B) имеет место равенство

$$B({}^{-1}B(x,y),z) = B(x,B^{-1}(y,z))\ ^{17)}$$

для любых $x,y,z \in Q$, то (Q,B) группа.

Доказательство.

В IP-лупах (Q,B) имеют место равенство

$${}^{-1}x = x^{-1}$$

для каждого $x \in Q$; эквивалентность

$$B(a,x) = b \Leftrightarrow x = B(a^{-1},b)$$

для любых $a,b,x \in Q$; и эеквивалентность

$$B(y,a) = b \Leftrightarrow y = B(b,a^{-1})$$

для любых $a,b,y \in Q$ [2-4].

Отсюда, ввиду определений

$$B^{-1}(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} B(x,z) = y; \text{ и}$$

$${}^{-1}B(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} B(z,y) = x$$

---

[16] см. лемму 8_1.

[17] см. равенство под GA3 из леммы 7.

для любых $x,y,z \in Q$, находим, что имеют место равенства

$${}^{-1}B(u,v) = B(u, v^{-1});\ \text{и}$$

$$B^{-1}(u,v) = B(u^{-1},v)$$

для любых $u,v \in Q$. Ввиду этых ревенств, равенство

$$B({}^{-1}B(x,y),z) = B(x,B^{-1}(y,z))$$

превращается в равенство

$$B(B(x,y^{-1}),z) = B(x,B(y^{-1},z))$$

для любых $x,y,z \in Q$. Отсюда, так как отображение

$$u \mapsto u^{-1}$$

подстановка множества Q, находим, что утверждение доказано.

ТЕОРЕМА 9. Пусть (Q,A) n-квазигруппа Бола, $n \in N\setminus\{1,2\}$. Тогда, если

(а) $B_{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} A(x,a_1^{n-2},y)$; и

(б) $B_{(b_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} A(x,b_1^{n-2},y)$

для любых $x,y \in Q$, где a_1^{n-2} и b_1^{n-2} любые фиксированные элементы множества Q, то $(Q,B_{(a_1^{n-2})})$ и $(Q,B_{(b_1^{n-2})})$ изоморфные группы. Притом, имеет место равенство:

(ц) $A(x,b_1^{n-2},y) = B_{(a_1^{n-2})}(x,B_{(a_1^{n-2})}(e_{\{1,n\}}(b_1^{n-2})^{-1},y))$,

где $e_{\{1,n\}}$ является $\{1,n\}$-нейтральной операцией n-квазигруппы (Q,A) 13).

Доказательство.

Ввиду леммы 7, леммы 8_1 и леммы 8_2, $(Q,B_{(a_1^{n-2})})$ и

$(Q,B_{(b_1^{n-2})})$ группы.

Далее, ввиду леммы 6_1, группы $(Q,B_{(a_1^{n-2})})$ и $(Q,B_{(b_1^{n-2})})$ изотопные. Таким образом, ввиду известной теоремы Алберта [2-4], $(Q,B_{(a_1^{n-2})})$ и $(Q,B_{(b_1^{n-2})})$ изоморфные группы.

Наконец, так как $(Q,B_{(a_1^{n-2})})$ группа, учитывая (а), (б), лемму 6_2 и доказательство леммы 8_2, находим, что имеет место равенство под (ц).

ТЕОРЕМА 10. Пусть (Q,B) группа и ε отображение множества Q^{n-2} в множество Q, $n \in N\backslash\{1,2\}$. Пусть, далее, для $n = 3$ ε подстановка множества Q, и для $n > 3$ (Q,ε) $(n-2)$-квазигруппа. Тогда, если

$$\text{(A)} \qquad A(x,b_1^{n-2},y) \overset{\text{деф}}{=\!=} B(x,B(\varepsilon(b_1^{n-2})^{-1},y))^{18)}$$

для любых $x,y,b_1^{n-2} \in Q$, то (Q,A) n-квазигруппа Бола и ε ее $\{1,n\}$-нейтральная операция.

Доказательство.

Так как (Q,B) группа а ε подстановка множества Q при $n = 3$ и (Q,ε) $(n-2)$-квазигруппа при $n > 3$, учитывая определение под (A), находим, что (Q,A) n-квазигруппа; [5].

Из (A), так как (Q,B) группа, непосредственно находим, что ε $\{1,n\}$-нейтральная операция n-квазигруппы (Q,A).

Далее, равенства под nB_L и nB_R из определения 2 запишем в следующем виде:

$$\text{(L)} \qquad A(A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},x)),z_1^{n-2},z) =$$

$$= A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-2},z))); \text{ и}$$

$$\text{(R)} \qquad A(z,z_1^{n-2},A(A(x,\overset{n-2}{x},y),\overset{n-2}{x},x)) =$$

$$= A(A(A(z,z_1^{n-2},x),\overset{n-2}{x},y),\overset{n-2}{x},x).$$

Учитывая определение под (A) и факт, что в группе имеет

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18) $= B(B(x,\varepsilon(b_1^{n-2})^{-1}),y)$.

место обобщенная ассоциативность, находим, что имеют место следующие цепи равенств:

$$(L_1)\quad A(A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},x)),z_1^{n-2},z) =$$

$$= B(B(x,B(\varepsilon(\overset{n-2}{x})^{-1},B(y,B(\varepsilon(\overset{n-2}{x})^{-1},x))),B(\varepsilon(z_1^{n-2})^{-1},z)) =$$

$$= B(B(B(B(B(B(x,\varepsilon(\overset{n-2}{x})^{-1}),y),\varepsilon(\overset{n-2}{x})^{-1}),x),\varepsilon(z_1^{n-2})^{-1}),z);$$

$$(L_2)\quad A(x,\overset{n-2}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-2},z))) =$$

$$= B(B(x,\varepsilon(\overset{n-2}{x})^{-1}),B(B(y,\varepsilon(\overset{n-2}{x})^{-1}),B(B(x,\varepsilon(z_1^{n-2})^{-1}),z))) =$$

$$= B(B(B(B(B(B(x,\varepsilon(\overset{n-2}{x})^{-1},y),\varepsilon(\overset{n-2}{x})^{-1}),x),\varepsilon(z_1^{n-2})^{-1}),z);$$

$$(R_1)\quad A(z,z_1^{n-2},A(A(x,\overset{n-2}{x},y),\overset{n-2}{x},x)) =$$

$$= B(B(z,\varepsilon(z_1^{n-2})^{-1}),B(B(B(x,\varepsilon(\overset{n-2}{x})^{-1}),y),B(\varepsilon(\overset{n-2}{x})^{-1},x))) =$$

$$= B(B(B(B(B(B(z,\varepsilon(z_1^{n-2})^{-1}),x),\varepsilon(\overset{n-2}{x})^{-1}),y),\varepsilon(\overset{n-2}{x})^{-1}),x);$$

$$(R_2)\quad A(A(A(z,z_1^{n-2},x),\overset{n-2}{x},y),\overset{n-2}{x},x) =$$

$$= B(B(B(z,B(\varepsilon(z_1^{n-2})^{-1},x)),B(\varepsilon(\overset{n-2}{x})^{-1},y)),B(\varepsilon(\overset{n-2}{x})^{-1},x)) =$$

$$= B(B(B(B(B(B(z,\varepsilon(z_1^{n-2})^{-1}),x),\varepsilon(\overset{n-2}{x})^{-1}),y),\varepsilon(\overset{n-2}{x})^{-1}),x)$$

для любых $x,y,z,z_1^{n-2} \in Q$.

Учитывая (L_1) и (L_2), находим, что имеет место равенство под (L) для любых $x,y,z,z_1^{n-2} \in Q$. Подобно, учитывая (R_1) и (R_2), находим, что имеет место равенство под (R) для любых $x,y,z,z_1^{n-2} \in Q$.

Теорема доказана.

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REZIME

n-KVAZIGRUPE BOLA

U radu se razmatraju n-kvazigrupe (Q,A), $n \in N\setminus\{1\}$, u kojima važe zakoni

$$A(A(\overset{n-1}{x},A(y,\overset{n-1}{x})),z_1^{n-1}) = A(\overset{n-1}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-1}))); \text{ i}$$

$$A(z_1^{n-1},A(A(\overset{n-1}{x},y),\overset{n-1}{x})) = A(A(A(z_1^{n-1},x),\overset{n-2}{x},y),\overset{n-1}{x}),$$

koji se pri $n = 2$ svode na levi i desni Bolov zakon [2-4]. Autor je ovakve n-kvazigrupe nazvao n-kvazigrupe Bola. Dokazano je da u (Q,A), $n \in N\setminus\{1\}$, postoji $(n-2)$-arna operacija $e_{\{1,n\}}$ koja zadovoljava uslov:

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(e_{\{1,n\}}(a_1^{n-2}),a_1^{n-2},x) = x \;\wedge$$
$$\wedge\; A(x,a_1^{n-2},e_{\{1,n\}}(a_1^{n-2})) = x);$$

$e_{\{1,n\}}$ se naziva $\{1,n\}$-neutralna operacija n-grupoida (Q,A), koju je autor uveo u radu pod [1]. (Na taj način, n-kvazigrupa Bola pri $n = 2$ jeste lupa Moufang [2-4].) Pri tom, $e_{\{1,n\}}$ za $n = 3$ jeste permutacija skupa Q, a pri $n > 3$, $(Q,e_{\{1,n\}})$ jeste $(n-2)$-kvazigrupa. Sledeća dva tvrđenja leže u osnovi konstrukcija n-kvazigrupa Bola ($n \in N\setminus\{1,2\}$): 1. Ako je (Q,A) n-kvazigrupa Bola, $n \in N\setminus\{1,2\}$, i $e_{\{1,n\}}$ je njena $\{1,n\}$-neutralna operacija, onda postoji grupa (Q,B) takva da je $A(x,b_1^{n-2},y) = B(x,B(e_{\{1,n\}}(b_1^{n-2})^{-1},y))$; i 2. Ako je (Q,B) grupa a ε permutacija skupa Q, ili je (Q,ε) $(n-2)$-kvazigrupa, $n \in N\setminus\{1,2,3\}$, onda je (Q,A), gde je $A(x,b_1^{n-2},y) \overset{def}{=} B(x,B(\varepsilon(b_1^{n-2})^{-1},y))$, n-kvazigrupa Bola, pri čemu je ε njena $\{1,n\}$-neutralna operacija.

SUMMARY

BOL'S n-QUASIGROUPS

In this paper n-quasigroups (Q,A), $n \in N \setminus \{1\}$, satisfying the laws

$$A(A(\overset{n-1}{x},A(y,\overset{n-1}{x})),z_1^{n-1}) = A(\overset{n-1}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-1})))$$

and

$$A(z_1^{n-1},A(A(\overset{n-1}{x},y),\overset{n-1}{x})) = A(A(A(z_1^{n-1},x),\overset{n-2}{x},y),\overset{n-1}{x})$$

are considered. For $n = 2$ these laws reduce to Bol's left and right law [2-4]. The author call such n-quasigroups Bol's n-quasigroups. It is proved that in (Q,A), $n \in N \setminus \{1\}$, there is an (n-2)-ary operation $e_{\{1,n\}}$ satisfying the following condition:

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(e_{\{1,n\}}(a_1^{n-2}),a_1^{n-2},x) = x \;\wedge$$

$$\wedge\; A(x,a_1^{n-2},e_{\{1,n\}}(a_1^{n-2})) = x);$$

$e_{\{1,n\}}$ is said to be a {1,n}-neutral operation of n-groupoid (Q,A), introduced by the author in the paper [1]. (Hence, for $n = 2$, a Bol's n-quasigroup is a Moufang's loop.) For $n = 3$, $e_{\{1,n\}}$ is a permutation of the set Q, and for $n > 3$, $(Q,e_{\{1,n\}})$ is an (n-2)-quasigroup. Constructions of Bol's n-quasigroups, $n \in N \setminus \{1,2\}$, are based on the following two propositions: 1. If (Q,A) is a Bol's n-quasigroup, $n \in N \setminus \{1,2\}$, and $e_{\{1,n\}}$ is it's {1,n}-neutral operation, then there is a group (Q,B) such that $A(x,b_1^{n-2},y) = B(x,B(e_{\{1,n\}}(b_1^{n-2})^{-1},y))$; and 2. If (Q,B) is a group, and either ε is a permutation of the set Q or (Q,ε) is an (n-2)-quasigroup, $n \in N \setminus \{1,2,3\}$, then for $A(x,b_1^{n-2},y) \underset{\text{def}}{=} B(x,B(\varepsilon(b_1^{n-2})^{-1}, y))$, (Q,A) is a Bol's n-qasigroup, where ε is it's {1,n}-neutral operation.

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О ДВУХ КЛАССАХ АЛГЕБР БЛИЗКИХ ГРУППАМ

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РЕЗЮМЕ

В работе определяются и рассматриваются классы алгебр среди которых, кроме групп, являются, например, клики [1] и полуклики [2]. Алгебры этых классов автор позволил себе назвать R-группами и L-группамы.

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ОПРЕДЕЛЕНИЕ 1_1. Пусть $\square$ бинарная операция в множестве G. Пустъ, далее, f подстановка множества G, удовлетворяющая условию

$$f \bullet f = I, \tag{O}$$

где $\bullet$ является умножением подстановок, а I единичная подстановка множества G. Объект $(G,\square,f)$ назовеи R-группой[1] тогда и только тогда, когда имеет место:

RG1 $(\forall a \in G)(\forall b \in G)(\forall c \in G)(a \square b)\square c = a \square f(fb \square c)$; и

RG2 $(\exists i \in G)(\exists j \in G)(\forall a \in G)(\exists\, {}^{\triangle}a \in G)(\exists a^{\triangle} \in G)(\exists\, {}^{\blacktriangle}a \in G)$

$$(\exists a^{\blacktriangle} \in G)(a \square i = a \wedge j \square a = fa \wedge {}^{\triangle}a \square a = i \wedge$$
$$\wedge\ a \square a^{\triangle} = i \wedge {}^{\blacktriangle}a \square a = j \wedge a \square a^{\blacktriangle} = j).$$

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[1] правой-группой.

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Примечание 1.

Если $f = I$, то речь идет о группах.

ЛЕММА 1. Пусть $\square$ бинарная операция в множестве G. Пусть, далее, f подстановка множества G, удовлетворяющая условию

$$(\mathrm{o}) \qquad f \bullet f = I,$$

где $\bullet$ умножение подстановок, а I единичная подстановка множества G. Тогда справедливо:

1° Если существуют $i,j \in G$ удовлетворяющие условию

$$(1) \qquad a \square i = a \wedge j \square b = fb \text{ для любых } a,b \in G\ ,$$

то имеет место:

(2) $fi = j$[1] ; и

2° не существуют больше чем один $i \in G$ и больше чем один $j \in G$ удовлетворяющие условию (1).

Доказательство.

а) Ввиду (1), находим, что имеет место:

$$j \square i = j \wedge j \square i = fi.$$

Отсюда получаем равенство (2).

б) Пусть существуют $i,j \in G$ и $\bar{i},\bar{j} \in G$ удовлетворяющие условиям:

$$a \square i = a \wedge j \square a = fa \quad \text{и}$$

$$b \square \bar{i} = b \wedge \bar{j} \square b = fb$$

для любых $a,b \in G$, т.е., ввиду (2), условиям

$$a \square i = a \wedge fi \square a = fa \qquad \text{и}$$

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[1] и: $fj = i$, ввиду (0).

$$b \square \bar{i} = b \wedge f\bar{i} \square b = fb$$

для любых $a, b \in G$. Отсюда, например, для $a = f\bar{i}$ и $b = i$, получаем, что имеют место равенства

$$f\bar{i} \square i = f\bar{i} \quad \text{и} \quad f\bar{i} \square i = fi,$$

т.е., что имеет место равенство

$$f\bar{i} = fi.$$

Так как f подстановка множества G, и утвреждение под б) доказано.

Учитывая определение 1_1 и лемму 1, находим что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 2. Пусть $(G, \square, f)$ R-группа. Тогда справедливо:

$\bar{1}$. если $i \in G$ и $j \in G$ удовлетворяют условию RG2, то имеет место:

(2) $fi = j$[1] ; и

$\bar{2}$. существуют в точности один $i \in G$ и в точности один $j \in G$ удовлетворяющие формуле под RG2.

УТВЕРЖДЕНИЕ 3. Пусть $(G, \square, f)$ R-группа. Тогда в $(G; \square)$ выполняются законы сокращения:

$$(\forall a \in G)(\forall b \in G)(\forall c \in G)(a \square c = b \square c \Rightarrow a = b) \quad \text{и}$$

$$(\forall a \in G)(\forall b \in G)(\forall c \in G)(c \square a = c \square b \Rightarrow a = b).$$

Доказательство.

а) Пусть $a, b, c \in G$ удовлетворяют условию $a \square c = b \square c$. Тогда, учитывая RG1, RG2 и утверждение 2, находим, что имеет место следующая цепь импликации:

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[1] и: $fj = i$, ввиду (0).

$$\begin{aligned} a \square c = b \square c &\Rightarrow (a \square c) \square (fc)^{◣} = (b \square c) \square (fc)^{◣} \\ &\Rightarrow a \square f(fc \square (fc)^{◣}) = b \square f(fc \square (fc)^{◣}) \\ &\Rightarrow a \square fj = b \square fj \\ &\Rightarrow a \square i = b \square i \\ &\Rightarrow a = b, \end{aligned}$$

т.е., что имеет место импликация:

$$a \square c = b \square c \Rightarrow a = b.$$

б) Пусть $a, b, c \in G$ удовлетворяют условию $c \square a = c \square b$. Тогда, учитывая RG1, RG2 и факт, что f подстановка множества G удовлетворяющая условию (O), находим, что имеет место следующая цепь импликации:

$$\begin{aligned} c \square a = c \square b &\Rightarrow f(c \square a) = f(c \square b) \\ &\Rightarrow f(ffc \square a) = f(ffc \square b) \\ &\Rightarrow {}^{◢}(fc) \square f(ffc \square a) = \\ &= {}^{◢}(fc) \square f(ffc \square b) \Rightarrow ({}^{◢}(fc) \square fc) \square a \\ &= ({}^{◢}(fc) \square fc) \square b \Rightarrow j \square a = j \square b \\ &\Rightarrow fa = fb \Rightarrow a = b, \end{aligned}$$

т.е., что имеет место импликация:

$$c \square a = c \square b \Rightarrow a = b.$$

В самом деле, утверждение доказано.

Учитывая лемму 1 и утверждение 3, находим, что имеет место:

ЛЕММА 4. Пусть $(G, \square, f)$ R-группа. Тогда множества ⊿, ◺, ◢ и ◣, определена следующим образом

$$\text{◿} \overset{\text{деф}}{=} \{\, a, {}^{\text{◿}}a) \mid a \in G\},$$

$$\text{◺} \overset{\text{деф}}{=} \{\, a, a^{\text{◺}}\,) \mid a \in G\},$$

$$\text{◢} \overset{\text{деф}}{=} \{\, a, {}^{\text{◢}}a) \mid a \in G\} \text{ и}$$

$$\text{◣} \overset{\text{деф}}{=} \{(a, a^{\text{◣}}\,) \mid a \in G\},$$

вляются унарными операциями в множестве G.

Ввиду леммы 1 и леммы 4 имеет место:

ТЕОРЕМА 5. Пусть $(G,\square,f)$ R-группа. Пусть, далее, ◿, ◺, ◢ и ◣ унарные операции в G, а i нульарная операция в множестве G. Тогда R-группе $(G,\square,f)$ однозначно соответствует объект $(G,\square,f,\text{◿},\text{◺},\text{◢},\text{◣},i)$ удовлетворяющий условиям:

VRG1 $(\forall a \in G)(\forall b \in G)(\forall c \in G)(a \square b) \square c = a \square f(fb \square c);$

VRG2 $(\forall a \in G)(a \square i = a \wedge fi \square a = fa);$ и

VRG3 $(\forall a \in G)({}^{\text{◿}}a \square a = i \wedge a \square a^{\text{◺}} = i \wedge {}^{\text{◢}}a \square a = fi \wedge$

$\wedge\ a \square a^{\text{◣}} = fi).$

И обратно.

Примечание 2.

Ввиду теоремы 5, объект $(G,\square,f,\text{◿},\text{◺},\text{◢},\text{◣},i)$, удовлетворяющий условиям VRG1 - VRG3, также, позволим себе назвать R-группой.

Ввиду монотонии и утверждения 3, имеет место:

ЛЕММА 6. Если $(G,\square,f)$ R-группа, то в $(G,\square)$ имеют место формулы:

(3_1) $(\forall a \in G)(\forall b \in G)(\forall c \in G)(a = b \Leftrightarrow a \square c = b \square c)$ и

(3_2) $(\forall a \in G)(\forall b \in G)(\forall c \in G)(a = b \Leftrightarrow c \square a = c \square b).$

УТВЕРЖДЕНИЕ 7_1. Пусть $(G,\square,f)$ R-группа. Тогда $(G,\square)$ квазигруппа и имеет место:

(4_1) $$a \square x = b \iff x = f({}^{\text{◢}}(fa) \square fb) \quad \text{и}$$

(4_2) $$y \square a = b \iff y = b \square (fa)^{\text{◣}}.$$

Доказательство.

а) Учитывая факт, что f подстановка множества G удовлетворяющая условию (O), RG1 и RG2, ввиду леммы 6, находим, что имеет место следующая цепь эквиваленции:

$$\begin{aligned} a \square x = b &\iff f(a \square x) = fb \\ &\iff f(ffa \square x) = fb \\ &\iff {}^{\text{◢}}(fa) \square f(ffa \square x) = {}^{\text{◢}}(fa) \square fb \\ &\iff ({}^{\text{◢}}(fa) \square fa) \square x = {}^{\text{◢}}(fa) \square fb \\ &\iff j \square x = {}^{\text{◢}}(fa) \square fb \\ &\iff fx = {}^{\text{◢}}(fa) \square fb \\ &\iff x = f({}^{\text{◢}}(fa) \square fb). \end{aligned}$$

Отсюда получаем, что имеет место эквиваленция:

(4_1) $$a \square x = b \iff x = f({}^{\text{◢}}(fa) \square fb).$$

Ввиду леммы 4 и факта, что (4_1) имеет место для любых $a,b \in G$, из (4_1) находим, что уравнение $a \square x = b$ однозначно разрешимо для любых $a,b \in G$.

б) Учитывая RG1, RG2 и утвреждение 2, ввиду леммы 6, находим, что имеет место следующая цепь эквиваленции:

$$\begin{aligned} y \square a = b &\iff (y \square a) \square (fa)^{\text{◣}} = b \square (fa)^{\text{◣}} \\ &\iff y \square f(fa \square (fa)^{\text{◣}}) = b \square (fa)^{\text{◣}} \end{aligned}$$

$$\Leftrightarrow y \,\square\, fj = b \,\square\, (fa)^{◣}$$

$$\Leftrightarrow y \,\square\, i = b \,\square\, (fa)^{◣}$$

$$\Leftrightarrow y = b \,\square\, (fa)^{◣}.$$

Отсюда получаем что имеет место эквиваленция:

(4_2) $$y \,\square\, a = b \Leftrightarrow y = b \,\square\, (fa)^{◣}.$$

Ввиду лемы 4 и факта, что (4_2) имеет место для любых $a, b \in G$, из (4_2) находим, что уравнение $y \,\square\, a = b$ однозначно разрешимо для любых $a, b \in G$.

Примечание 3.

Учитывая утверждение 7_1, находим, что ⊿, ◺, ◢ и ◣ являются подстановками множества G. Притом, если ⊿ = ◺ (◢ = ◣), то ⊿ (◢) удовлетворяет условию ⊿ • ⊿ = I (◢ • ◢ = I).

УТВЕРЖДЕНИЕ 7_2. Пуст □ бинарная операция в множестве G. Пусть, далее, f подстановка множества G удовлетворяющая условию

(о) $$f \bullet f = I,$$

где • умножение подстановок а I единичная подстановка. Тогда, если имеет место RG1 и $(G, \square)$ является квазигруппой, то $(G, \square, f)$ является R-группой.

Доказательство.

а) Пусть a любой фиксированый элемент множества G. Так как $(G, \square)$ квазигруппа, то существует в точности один $i_a \in G$ такой, что имеет место

(а) $$a \,\square\, i_a = a.$$

Притом, для любого $b \in G$ существует в точности один $y \in G$ такой, что имеет место

(б) $$b = y \,\square\, fa.$$

Далее, учитывая RG1, (о), (а) и (б), находим, что имеет место следующая цепь равенств:

$$b \square i_a = (y \square fa) \square i_a = y \square f(ffa \square i_a) =$$

$$= y \square f(a \square i_a) = y \square fa =$$

$$= b.$$

Таким образом, мы доказали, что имеет место формула:

(в) $$(\exists! i \in G)(\forall x \in G) x \square i = x.$$

б) Пусть а любой фиксированый элемент множества G. Так как $(G, \square)$ квазигруппа, то существует в точности один $j_a \in G$ такой, что имеет место

(а̄) $$j_a \square a = fa.$$

Притом, для любого $b \in G$ существует в точности один $x \in G$ такой, что имеет место

(б̄) $$b = f(fa \square x)$$ 1).

Далее, учитывая RG 1, (а̄) и (б̄), находим, что имеет место следующая цепь равенств:

$$j_a \square b = j_a \square f(fa \square x) = (j_a \square a) \square x$$

$$= fa \square x = fb.$$

Таким образом, мы доказали, что имеет место формула:

(в̄) $$(\exists! j \in G)(\forall y \in G) j \square y = fy.$$

в) Ввиду (в) и (в̄), находим, что имеют место равенства

---

1) $\Leftrightarrow fb = fa \square x$.

$$j \square i = j \quad \text{и} \quad j \square i = fi,$$

т.е. равенство

(г) $$j = fi^{1)}\,.$$

г) Учитывая (в), ($\bar{\text{в}}$) и (г), находим, что имеет место VRG2 из теоремы 5.

д) Ввиду предположения, что $(G,\square)$ квазигруппа, имеет место и VRG3 из теоремы 5.

Наконец, учитывая теорему 5, находим, что утверждение доказано.

Непосредственным следствием утверждения 7_1 и утверждения 7_2 является следующая:

ТЕОРЕМА 8. $(G,\square,f)$ является R-группой тогда и только тогда, когда имеет место RG1 и $(G,\square)$ является квазигруппой.

Нахождением решений уравнений

$$(a \square b) \square x = i, \quad y \square f(a \square b) = i,$$

$$(a \square b) \square u = j \quad \text{и} \quad v \square f(a \square b) = j,$$

полученых различними способами, ввиду утверждения 7_1, находим, что имеет место следующее утверждение:

ТЕОРЕМА 9. Пусть $(G,f,\text{◿},\text{◺},\text{◢},\text{◣},i)$ R-группа[2]. Тогда имеют место равенства:

(5_1) $$(a \square b)^{\text{◺}} = f((^{\text{◢}}b \square f^{\text{◢}}(fa)) \square fi);$$

(5_2) $$^{\text{◿}}(f(a \square b)) = (i \square (fb)^{\text{◣}}) \square a^{\text{◣}}\ ;$$

(5_3) $$(a \square b)^{\text{◣}} = f(^{\text{◢}}b \square f(^{\text{◢}}(fa))) \text{ и}$$

---

[1] Ввиду (о), и: $fj = i$.

[2] См. примечание 2.

$$(5_4) \qquad {}^{◢}(f(a \square b)) = f((fb)^{◣}) \square a^{◺}$$

для любых $a, b \in G$.

Примечание 4.

Если $f = I$, то $(G, \square, ^{-1}, i)$, где $-1 = ◿ = ◺ = ◢ = ◣$, является группой. Учитывая этот факт, мы находим, что каждое из равенств (5_1)-(5_4) сводится к равенству

$$(5) \qquad (a \square b)^{-1} = b^{-1} \square a^{-1}$$

для любых $a, b \in G$.

ТЕОРЕМА 10. Пусть $(G, \square, f)$ R-группа. Тогда квазигруппа, $(G, \square)$ изотопна[1] некоторой группе $(G, *)$.

Доказательство.

Положим:

$$(6_1) \qquad a * b \overset{\text{деф}}{=\!=} a \square fb$$

для любых $a, b \in G$. Ввиду (o), равенство под (6_1) эквивалентно равенству

$$(6_2) \qquad a \square b = a * fb$$

для любых $a, b \in G$.

Используя (6_1) и (6_2) в RG1, находим, что имеет место равенство

$$(a * fb) * fc = a * ff(fb * fc)$$

для любых $a, b, c \in G$. Отсюда, ввиду факта, что f подстановка множества G удовлетворяющая (o), находим, что $(G, *)$ является полугруппой.

Далее, ввиду (6_1), $(G, \square)$ и $(G, *)$ изотопны. Отсюда, учитывая утверждение 7_1, находим, что $(G,*)$ является и квазигруппой[1].

---

[1] например: [3-5].

Таким образом, мы доказали, что $(G,*)$ является группой, изотопной квазигруппе $(G,\square)$.

Подобным способом доказивается следующее утверждение:

УТВЕРЖДЕНИЕ 11. Пусть $(G,*)$ группа, а f подстановка множества G удовлетворяющая условию (o)[2]. Пусть, далее, имеет место

$$a \square b \overset{\text{деф}}{=\!=} a * fb$$

для любых $a,b \in G$. Тогда $(G,\square,f)$ является R-группой.

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	6	5	2	1	4
4	4	5	6	1	2	3
5	5	4	1	6	3	2
6	6	3	2	5	4	1

Табл. 1_1

□	1	2	3	4	5	6
1	3	4	1	2	6	5
2	4	3	2	1	5	6
3	5	2	3	6	4	1
4	6	1	4	5	3	2
5	1	6	5	4	2	3
6	2	5	6	3	1	4

Табл. 1_2

На табл. 1_1 изображена группа $(\{1,\ldots,6\},*)$ изоморфна группе подстановок множества $\{a,b,c\}$, т.е. группе $(\{a,b,c\}!,\circ)$[3]. На табл. 1_2 изображена квазигруппа $(\{1,\ldots,6\},\square)$, где

$$x \square y \overset{\text{деф}}{=\!=} x * fy \quad \text{и}$$

$$f \overset{\text{деф}}{=\!=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}.$$

---

[1] Изотоп квазигруппы и сам является квазигруппой; например [3-5].

[2] см. определение 1_1.

[3] $1 \leftrightarrow \left(\begin{smallmatrix} a & b & c \\ a & b & c \end{smallmatrix}\right)$, $2 \leftrightarrow \left(\begin{smallmatrix} a & b & c \\ a & c & b \end{smallmatrix}\right)$, $3 \leftrightarrow \left(\begin{smallmatrix} a & b & c \\ b & c & a \end{smallmatrix}\right)$, $4 \leftrightarrow \left(\begin{smallmatrix} a & b & c \\ b & a & c \end{smallmatrix}\right)$, $5 \leftrightarrow \left(\begin{smallmatrix} a & b & c \\ c & a & b \end{smallmatrix}\right)$, $6 \leftrightarrow \left(\begin{smallmatrix} a & b & c \\ c & b & a \end{smallmatrix}\right)$.

Так как f удовлетворяет условию (o)[1], ввиду утверждения 11, $(\{1,\dots,6\},□,f)$ является R-группой. Притом имеет место:

$$i = 3,\ j = fi = f3 = 1^{2)};$$

$$◿ = \{(1,1);(2,2);(3,3);(4,6);(5,4);(6,5)\};$$

$$◺ = \{(1,1);(2,2);(3,3);(4,5);(5,6);(6,4)\};$$

$$◢ = \{(1,5);(2,4);(3,1);(4,2);(5,6);(6,3)\};\ \text{и}$$

$$◣ = \{(1,3);(2,4);(3,6);(4,2);(5,1);(6,5)\}.$$

Заметим, что в рассматриваемой R-группе имеет место:

$$i \neq j \ \text{ и } \ |\{◿, ◺, ◢, ◣\}| = 4.$$

Таким образом, справедливо следующее утверждение:

УТВЕРЖДЕНИЕ 12. Существует R-группа $(G,□,f,◿,◺,◢,◣,i)^{3)}$ в которой справедливо:

$$fi \neq i \ \text{ и } \ |\{◿, ◺, ◢, ◣\}| = 4.$$

ТЕОРЕМА 13_1. Пусть $(G,□,f,◿,◺,◢,◣,i)$ R-группа. Пусть, далее,

$$(6_1) \qquad a * b \overset{\text{деф}}{=} a \;□\; fb^{\,4)}$$

для любых $a,b \in G$. Тогда, если

$$(7) \qquad ◿ = ◺ = ◢ = ◣ = I,$$

то $(G,*,f,i)$ группа, где i единица а $fa \in G$ обратный элемент элемента $a \in G$ для любого $a \in G$.

---

[1] см. определение 1_1.

[2] $i = 3$ правая единица в $(G,□)$, а $j = 1$ единица в $(G,*)$!

[3] см. примечание 2.

[4] $\Leftrightarrow a \;□\; b = a * fb$; (6_2).

Доказательство.

а) Ввиду доказательства теоремы 10, $(G,*)$ группа.

б) Ввиду (7), имеют место равенства:

$$a \square a = i \quad \text{и} \quad a \square a = fi$$[1]

для любого $a \in G$. Отсюда следует, что

$$fi = i.$$

в) Ввиду б) $(fi = i)$, имеет место

$$a \square i = a \quad \text{и} \quad i \square a = fa$$

для любого $a \in G$. Отсюда, учитывая (6_1), находим, что имеет место

$$a * fi = a \quad \text{и} \quad i * fa = fa$$

для любого $a \in G$. Таким образом, ввиду б) $(fi = i)$ и факта, что f подстановка множества G, i является единицей группы $(G,*)$.

г) Ввиду (7), имеет место равенство

$$a \square a = i$$

для любого $a \in G$. Отсюда, ввиду (6_1), находим, что

$$a * fa = i$$

для любого $a \in G$, т.е., что fa правый обратный элемент элемента $a \in G$ для любого $a \in G$. Таким образом, ввиду факта, что $(G,*)$ группа, мы доказали, что fa обратный элемент элемента $a \in G$ для любого $a \in G$.

Теорема доказана.

Подобным способом доказывается и следующее утверждение:

ТЕОРЕМА 13_2. Пусть $(G,*,f,i)$ группа, где i единица а f унарная операция - взятие обратного элемента. Пусть, далее,

(6_2) $$a \square b = a * fb$$

---

[1] $fi = j$; см. утверждение 2.

для любых $a,b \in G$. Тогда $(G,\square,f,◿,◺,◢,◣,i)$ R-группа в которой имеют место равенства

$$◿ = ◺ = ◢ = ◣ = I^{1)} .$$

Учитывая равенства под (7) из теоремы 13_1 в равенстве под (5_4) из теоремы 9, находим, что имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 14. Если в R-группе $(G,\square,f,◿,◺,◢,◣,i)$ имеют место равенства

$$◿ = ◺ = ◢ = ◣ = I^{1)},$$

то имеет место равенство

$$f(a \square b) = b \square a \tag{8}$$

для любых $a,b \in G$.

Притом, обратное не имеет место. На табл. 1_1 изображена группа $(\{1,\dots,6\},*)$ изоморфна группе подстановок множества $\{a,b,c\}$, т.е. группе $(\{a,b,c\}!,\bullet)$. На табл. 1_3 изображена квазигруппа $(\{1,\dots,6\},\square)$, где

□	1	2	3	4	5	6
1	1	2	3	6	5	4
2	2	1	4	5	6	3
3	3	6	5	4	1	2
4	4	5	6	3	2	1
5	5	4	1	2	3	6
6	6	3	2	1	4	5

Табл. 1_3

$$x \square y \overset{\text{деф}}{=\!=} x * fy \quad \text{и}$$

$$f \overset{\text{деф}}{=\!=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}.$$

Так как f удовлетворяет условию $(o)^{2)}$, ввиду утверждения 11, $(\{1,\dots,6\},\square,f)$ является R-группой. Непосредствено из табл. 1_3 находим, что имеет место:

$$i = 1, \quad j = fi = f1 = 1; \quad \text{и}$$

---

1) равенства под (7) из теоремы 13_1.

2) см. определение 1_1.

◿ = ◺ = ◢ = ◣ = $\{(1,1);(2,2);(3,5);(4,6);(5,3);(6,4)\} \neq I$. Притом, проверкой находим, что имеет место равенство:

$$f(a \square b) = b \square a$$

для любых $a, b \in G$. Таким образом имеет место:

УТВЕРЖДЕНИЕ 15. Существует R-группа (G,□,f,◿,◺,◢,◣,i) неудовлетворяющая условию под (7) из теоремы 13_1 в которой имеет место равенство

$$f(a \square b) = b \square a \tag{8}$$

для любых $a, b \in G$.

Притом имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 16. Пусть в R-группе (G,□,f,◿,◺,◢,◣,i) имеет место равенство

$$f(a \square b) = b \square a \tag{8}$$

для любых $a, b \in G$. Тогда справедливо:

$\bar{1}$. $fi = i$; и

$\bar{2}$. ◿ = ◺ = ◢ = ◣ .

Доказательство.

Из равенства под (8) для $a = b = i$ вытекает равенство

$$f(i \square i) = i \square i.$$

Отсюда, ввиду VRG2 из теоремы 5, вытекает равенство

$$fi = i.$$

Далее, из

$^{◿}a \square a = i$ и $^{◢}a \square a = i$[1],

ввиду (8) и $fi = i$, вытекают равенства

---

[1] мы уже доказали, что $j = fi = i$; $\bar{1}$.

$$a \,\square\, ◿a = i \quad \text{и} \quad a \,\square\, ◢a = i\,.$$

Таким образом, ввиду утверждения 3, мы доказали, что имеет место равенства:

$$◿ = ◺ = ◢ = ◣\,.$$

Утверждение доказано.

Притом, обратное не имеет место. На табл. 1_1 изображена группа ({1,...,6},*) изоморфна группе подстановок множества {a,b,c}, т.е. группе ({a,b,c}!,o). На табл. 1_4 изображена квазигруппа ({1,...,6},□), где

□	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	3	6	1	4	5	2
4	4	5	2	3	6	1
5	5	4	3	2	1	6
6	6	3	4	1	2	5

Табл. 1_4

$$x \,\square\, y \overset{\text{деф}}{=\!=} x * fy \quad \text{и}$$

$$f \overset{\text{деф}}{=\!=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 & 3 & 4 \end{pmatrix}.$$

Так как f удовлетворяет условию (o), ввиду утверждения 11, ({1,......,6},□,f) является R-группой. Непосредствено из табл. 1_4 находим, что имеет место:

$$i = 1, \quad j = fi = f1 = 1;$$

$$◿ = ◺ = ◢ = ◣ = \{(1,1);(2,2);(3,3);(4,6);(5,5);(6,4)\}; \text{ и}$$

$$f(3 \,\square\, 4) = f4 = 6 \neq 4 \,\square\, 3 = 2.$$

Таким образом имеет место:

УТВЕРЖДЕНИЕ 17. Существует R-группа (G,□,f,◿,◺,◢,◣,i) в которой не имеет место формула

$$(\forall a \in G)(\forall b \in G) f(a \,\square\, b) = b \,\square\, a$$

и имеют место $\bar{1}$ и $\bar{2}$ из утверждения 16.

Притом имеет место следующее утверждение:

УТВЕРЖДЕНИЕ 18. Пусть в R-группе $(G, \square, f, ⊿, ◺, ◢, ◣, i)$ имеет место равенство

$$f(a \square b) = fa \square fb \tag{9}$$

для любых $a, b \in G$[1]. Тогда справедливо:

$\bar{1}$. $fi = i$; и

$\bar{2}$ $⊿ = ◺ = ◢ = ◣$.

Доказательство.

а) Ввиду предположения, что f автоморфизм квазигруппы $(G, \square)$, имеет место равенство

$$f(i \square i) = fi \square fi.$$

Отсюда, ввиду VRG2 из теоремы 5, находим, что имеет место равенство

$$fi = fi \square fi.$$

Отсюда, как так, ввиду VRG2 из теоремы 5, имеет место равенство

$$fi = fi \square i,$$

учитывая утверждение 3, находим, что

$$fi = i.$$

б) Ввиду равенства $i = fi$ $(= j)$, на основании утверждения 3, имеют место равенства

$$⊿ = ◢ \quad \text{и} \quad ◺ = ◣.$$

в) Ввиду VRG2 из теоремы 5 и равенства $fi = i$, имеют место равенства

$$({}^{⊿}a \square a) \square (fa)^{◺} = i \square (fa)^{◺} = f((fa)^{◺}) \quad \text{и}$$

---

[1] т.е.: пусть f автоморфизм квазигруппы $(G, \square)$.

$$◿a \square f(fa \square (fa)^{◺}) = ◿a \square i = ◿a.$$

Отсюда, ввиду VPG1 из теоремы 5, находим, что имеет место равенство

$$f((fa)^{◺}) = ◿a,$$

т.е., ввиду (о), равенство

$$(fa)^{◺} = f(◿a).$$

г) Ввиду предположения, что f автоморфизм квазигруппы $(G,\square)$ и равенства $fi = i$, находим, что имеют место равенства

$$i = fi = f(◿a \square a) = f(◿a) \square fa,$$

т.е. равенство:

$$f(◿a) \square fa = i.$$

Отсюда, ввиду равенства доказаного под в), находим, что имеет место равенство:

$$(fa)^{◺} \square fa = i.$$

Отсюда, как так, ввиду VRG3, имеет место и равенство

$$fa \square (fa)^{◺} = i,$$

учитывая факт, что f подстановка множества G, находим, что

$$◿ = ◺.$$

Наконец, отсюда, ввиду равенств доказаных под б), находим, что

$$◿ = ◺ = ◢ = ◣.$$

Утверждение доказано.

УТВЕРЖДЕНИЕ 19. Пусть $(G,\square,f,◿,◺,◢,◣,i)$ R-группа. Тогда, если

(10) $$fi = i,$$

то имеет место:

(10_1) $$(fa)^{◺} = f({}^{⊿}a);$$

(10_2) $${}^{⊿}(fa) = f(a^{◺}); \quad \text{и}$$

(10_3) $$(a \square b)^{◺} = f({}^{⊿}b \square a^{◺})^{1)},$$

для любых $a, b \in G$.

Доказательство.

а) Равенство под (10_1) доказано в рамках доказательства утверждения 18 под в).

б) Так как f подстановка множества G, удовлетворяющая условию (о), мы из равенства под (10_1) получаем равенство

$$a^{◺} = f({}^{⊿}(fa))$$

для любого $a \in G$. Отсюда, ввиду факта, что f подстановка множества G удовлетворяющая условию под (о), находим, что имеет место равенство под (10_2) для любого $a \in G$.

в) Равенство под (10_3) является частным случаем каждого из равенств под (5_1)-(5_4) из теоремы 9. В самом деле, ввиду справедливости равенств под (10), (10_1), (10_2), так как при $fi = i$ имеют место равенства $⊿ = ◢$ и $◺ = ◣$, каждое из равенств под (5_1)-(5_4) превращается в равенство под (10_3) для любого $a \in G$.

На табл. 1_1 изображена группа $(\{1,\dots,6\},*)$, изоморфная группе подстановок множества $\{a,b,c\}$, т.е. группе $(\{a,b,c\}!, \circ)$. На табл. 1_5 изображена квазигруппа $(\{1,\dots,6\},\square)$, где

$$x \square y \overset{\text{деф}}{=} x * fy \quad \text{и}$$

$$f \overset{\text{деф}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 6 & 5 & 4 & 3 \end{pmatrix}.$$

---

[1] Если $fi = i$, то $◢ = ⊿$ и $◣ = ◺$.

□	1	2	3	4	5	6
1	1	2	6	5	4	3
2	2	1	5	6	3	4
3	3	6	4	1	2	5
4	4	5	3	2	1	6
5	5	4	2	3	6	1
6	6	3	1	4	5	2

Табл. 1_5

Так как f удовлетворяет условию (о), ввиду утверждения 11, ({1,...,6},□,f) является R-группой. Непосредствено из табл. 1_5 находим, что имеет место:

i = 1, fi = 1 (= j);

⊿ = {(1,1);(2,2);(3,6);(4,3);(5,4);(6,5)};

◺ = {(1,1);(2,2);(3,4);(4,5);(5,6);(6,3)}; и

◢ = ⊿ ≠ ◣ = ◺.

Таким образом имеет место:

УТВЕРЖДЕНИЕ 20. Существует R-группа (G,□,f,⊿,◺,◢,◣,i) в которой имеет место

fi = i; и

◢ = ⊿ ≠ ◺ = ◣.

Если имеют место равенства ⊿ = ◺ = ◢ = ◣, то имеет место равенство fi = i. Отсюда, ввиду утверждения 19, находим, что имеет место утверждение:

УТВЕРЖДЕНИЕ 21. Пусть (G,□,f,⊿,◺,◢,◣,i) R-группа. Тогда, если

(11) ⊿ = ◺ = ◢ = ◣,

то имеет место:

(11_1) fi = i;

(11_2) (fa)$^{◺}$ = f(a$^{◺}$)[1] ; и

---

[1] т.е.: ◺ • f = f • ◺. Притом, смотри примечание 3.

(11_3) $$(a \square b)^{◺} = f(b^{◺} \square a^{◺}),$$

для любых $a, b \in G$.

Учитывая утверждение 16 и утверждение 21, находим, что имеет место:

ТЕОРЕМА 22. Если в R-группе $(G, \square, f, ◿, ◺, ◢, ◣, i)$, справедлива формула

VRG4[1] $(\forall a \in G)(\forall b \in G) f(a \square b) = b \square a$[2],

тогда имеет место:

$\bar{1}$. $fi = i$;

$\bar{2}$. $◿ = ◺ = ◢ = ◣$; и

$\bar{3}$. $(\forall a \in G)(\forall b \in G)(a \square b)^{◺} = a^{◺} \square b^{◺}$

Учитывая утверждение 18 и утверждение 21, находим, что имеет место:

ТЕОРЕМА 23. Если в R-группе $(G, \square, f, ◿, ◺, ◢, ◣, i)$ справедлива формула

VRG5[1] $(\forall a \in G)(\forall b \in G) f(a \square b) = fa \square fb$[3],

тогда имеет место:

$\bar{\bar{1}}$ $fi = i$;

$\bar{\bar{2}}$ $◿ = ◺ = ◢ = ◣$; и

$\bar{\bar{3}}$ $(\forall a \in G)(\forall b \in G)(a \square b)^{◺} = (fb)^{◺} \square (fa)^{◺}$.

Клика введена в [1] как объект $(G, \square, f)$ (где $\square : G^2 \to G$, $f \in G!$, $f \bullet f = I$ и $f \neq I$) удовлетворяющий условиям:

C1 $(\forall a \in G)(\forall b \in G)(\forall c \in G)(a \square b) \square c = a \square (b \square fc)$;

---

[1] см. теорему 5.

[2] см. (8) из утверждения 16.

[3] см. (9) из утверждения 18.

C2 $(\forall a \in G)(\forall b \in G)f(a \circ b) = b \circ a$$^{1)}$;

C3 $(\forall a \in G)(\forall b \in G)f(a \circ b) = fa \circ fb$$^{2)}$; и

C4 $(\forall a \in G)(\forall b \in G)(\exists x \in G)a \circ x = b$.

Притом, в клике имеет место:

ЛЕММА 24_1.[1] Если $(G,\circ,f)$ клика, то $(G,\circ)$ квазигруппа.

Полуклика введена в [2] как обобщение клики. В самом деле, полуклика объект $(G,\circ,f)$ (где $\circ : G^2 \to G$, $f \in G!$, $f \bullet f = I$ и $f \neq I$) удовлетворяющий условиям:

SC1 $(\forall a \in G)(\forall b \in G)(\forall c \in G)(a \circ b) \circ c = a \circ f(fb \circ c)$$^{3)}$;

SC2 $(\forall a \in G)(\forall b \in G)f(a \circ b) = b \circ a$$^{1)}$; и

SC3 $(\forall a \in G)(\forall b \in G)(\exists x \in G)a \circ x = b$.

Притом, в полуклике имеет место:

ЛЕММА 24_2.[2] Если $(G,\circ,f)$ полуклика, то $(G,\circ)$ квазигруппа.

Очевидно, что клика полуклика удовлетворяющая условию VRG5 из теоремы 23. Притом, ввиду теоремы 8, леммы 24_1 и леммы 24_2, находим что имеет место:

УТВЕРЖДЕНИЕ 25. а) $(G,\circ,f)$ является полукликой тогда и только тогда, когда $(G,\circ,f)$ R-группа удовлетворяющая условию VRG4 и $f \neq I$. б) $(G,\circ,f)$ является кликой тогда и только тогда, когда $(G,\circ,f)$ R-группа удовлетворяющая условию VRG4, условию VRG5 и $f \neq I$.

Таким образом, утверждения 13-16 и 22, в самом деле, относятся к полукликам. Притом, теорема 13_2, в соответствующем виде, рассмотрена уже в [2].

---

1) VRG4 из теоремы 22.

2) VRG5 из теоремы 23.

3) RG1 из определения 1_1 и VRG1 из теоремы 5.

На табл. 1_2 изображена R-группа удовлетворяющая условиям:

$i \neq j$ и $|\{$◿, ◺, ◢, ◣$\}| = 4$.

В утверждениях 16 и 18 речь идет о двух достаточных условиях для справедливости равенств:

$j = i$ и ◿ = ◺ = ◢ = ◣.

Притом, на табл. 1_3 и 1_4 изображены соответствующие R-группы. На табл. 1_5 изображена R-группа удовлетворяющая условиям:

$j = i$ и ◿ = ◢ ≠ ◺ = ◣.

На табл. 1_6 изображена R-группа удовлетворяющая условиям:

$j \neq i$ и ◿ = ◺ ≠ ◢ = ◣.

Притом, имеет место:

$$a \;\square\; b \overset{\text{деф}}{=} a * fb,\; f \overset{\text{деф}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 5 & 4 \end{pmatrix} \text{ и}$$

$(\{1,\ldots,6\},*)$ изображена на табл. 1_1.

□	1	2	3	4	5	6
1	2	1	3	6	5	4
2	1	2	4	5	6	3
3	6	3	5	4	1	2
4	5	4	6	3	2	1
5	4	5	1	2	3	6
6	3	6	2	1	4	5

Табл. 1_6

□	1	2	3	4	5	6
1	2	1	3	4	5	6
2	1	2	4	3	6	5
3	6	3	5	2	1	4
4	5	4	6	1	2	3
5	4	5	1	6	3	2
6	3	6	2	5	4	1

Табл. 1_7

На табл. 1_7 изображена R-группа удовлетворяющая условиям:

$j \neq i$ $(i = 2,\ j = 1)$, ◿ ≠ ◺ и ◢ = ◣.

Притом, имеет место:

$$a \;\square\; b \overset{\text{деф}}{=} a * fb,\quad f \overset{\text{деф}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix} \text{ и}$$

$(\{1,\ldots,6\},*)$ изображена на табл. 1_1.

□	1	2	3	4	5	6
1	2	1	4	3	6	5
2	1	2	3	4	5	6
3	6	3	2	5	4	1
4	5	4	1	6	3	2
5	4	5	6	1	2	3
6	3	6	5	2	1	4

Табл. 1_8

На табл. 1_8 изображена R-группа удовлетворяющая условиям:

$$i \neq j \quad (i = 2,\ j = 1),$$

◿ = ◺ и ◢ ≠ ◣.

Притом, имеет место:

$$a \,\square\, b \overset{\text{деф}}{=} a * fb\ ,$$

$$f \overset{\text{деф}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} \text{ и } (\{1,\ldots,6\},*)$$

изображена на табл. 1_1.

Непосредствено находим, что квазигруппа $(\{1,\ldots,6\},\square)$ изображена на табл. 1_8 является лупой, где единица $i = 2$. Отсюда, ввиду теоремы 10, на основании теоремы Алберта,[1] находим, что $(\{1,\ldots,6\},\square)$ является группой (изоморфной группе $(\{1,\ldots,6\},*)$ изображеной на табл. 1_1)[2] . Подобним способом, находим, что имеет место следующее утверждение:

ТЕОРЕМА 26. Пусть $(G,\square,f)$ R-группа. Тогда, если $(G,\square)$ обладает левой единицей, то $(G,\square)$ является группой.

* *

ОПРЕДЕЛЕНИЕ 1_2. Пусть ♦ бинарная операция в множестве G . Пусть, далее, f подстановка множества G, удовлетворяющая условию

$$\text{(o)} \qquad f \bullet f = I,$$

где • является умножением подстановом, а I единичная подстановка множества G. Объект (G,♦,f) назовем L-группой[3] тогда и только тогда, когда имеет место:

---

[1] например: [3-5].

[2] Притом: $f \neq I$.

[3] левой-группой.

LG1 $(\forall a \in G)(\forall b \in G)(\forall c \in G) f(a \blacklozenge fb) \blacklozenge c = a \blacklozenge (b \blacklozenge c)$; и

LG2 $(\exists i \subset G)(\exists j \in G)(\forall a \in G)(\exists\, {}^{\leftarrow\square}a \in G)(\exists a^{\square\rightarrow} \in G)(\exists\, {}^{\leftarrow\blacksquare}a \in G)$

$$(\exists a^{\blacksquare\rightarrow} \in G)(i \blacklozenge a = a \wedge a \blacklozenge j = fa \wedge {}^{\leftarrow\square}a \blacklozenge a = i \wedge$$
$$\wedge\, a \blacklozenge a^{\square\rightarrow} = i \wedge {}^{\leftarrow\blacksquare}a \blacklozenge a = j \wedge a \blacklozenge a^{\blacksquare\rightarrow} = j).$$

Примечание 1'.

Если $f = I$, то речь идет о группах.

Пусть $(G, \blacklozenge, f)$ L-группа. Пусть, далее,

(к) $$a \square b \stackrel{\text{деф}}{=\!=} b \blacklozenge a$$

(для любых $a, b \in G$. Тогда LG1 и LG2 превращаются, в том же порядке, в:

(а) $(\forall a \in b)(\forall b \in G)(\forall c \in G)(c \square b) \square a = c \square f(fb \square a)$; и

(б) $(\exists i \in G)(\exists j \in G)(\forall a \in G)(\exists\, {}^{\leftarrow\square}a \in G)(\exists a^{\square\rightarrow} \in G)($

$$(\exists\, {}^{\leftarrow\blacksquare}a \in G)(\exists a^{\blacksquare\rightarrow} \in G)(a \square i = a \wedge j \square a = fa \wedge$$
$$\wedge\, a \square {}^{\leftarrow\square}a = i \wedge a^{\square\rightarrow} \square a = i \wedge a \square {}^{\leftarrow\blacksquare}a = j \wedge a^{\blacksquare\rightarrow} \square a = j).$$

Формула под (а), в самом деле, является формулом под RG1 из определения 1_1. Если, далее, в формуле под (б)

$${}^{\leftarrow\square}a,\ a^{\square\rightarrow},\ {}^{\leftarrow\blacksquare}a \text{ и } a^{\blacksquare\rightarrow},$$

в том же порядке, обозначим через

$$a^{\triangle},\ {}^{\triangle}a,\ a^{\blacktriangle} \text{ и } {}^{\blacktriangle}a,$$

то формула под (б) превращется в формулу под RG2 из определения 1_1. Таким образом, $(G, \square, f)$ R-группа. Подобним способом, находим, что если $(G, \square, f)$ R-группа и имеет место

$(\bar{\text{к}})$ $$a \blacklozenge b \stackrel{\text{деф}}{=\!=} b \square a$$

для любых $a,b \in G$, то $(G,\blacklozenge,f)$ является L-группой.

В самом деле, мы доказали, что имеет место следующее утверждение:

ТЕОРЕМА 27. а) Если $(G,\blacklozenge,f)$ L-группа и имеет место

(к) $$a \,\square\, b \overset{\text{деф}}{=\!=} b \blacklozenge a$$

для любых $a,b \in G$, то $(G,\square,f)$ R-группа; и

б) Если $(G,\square,f)$ R-группа и имеет место

$(\bar{к})$ $$a \blacklozenge b \overset{\text{деф}}{=\!=} b \,\square\, a$$

для любых $a,b \in G$, то $(G,\blacklozenge,f)$ L-группа.

Учитывая теорему 27 и ее доказательство, находим, что имеют место, например, следующие два утверждения:

ТЕОРЕМА 5´. Пусть $(G,\blacklozenge,f)$ L-группа. Пусть, далее, $\leftarrow\square$, $\square\rightarrow$, $\leftarrow\blacksquare$ и $\blacksquare\rightarrow$ унарные операции в G, а i нульарная операция в множестве G. Тогда L-группе $(G,\blacklozenge,f)$ однозначно соответствует объект $(G,\blacklozenge,f,\leftarrow\square,\square\rightarrow,\leftarrow\blacksquare,\blacksquare\rightarrow,i)$ удовлетворяющий условиям:

VLG1 $(\forall a \in G)(\forall b \in G)(\forall c \in G)f(a \blacklozenge fb) \blacklozenge c = a \blacklozenge (b \blacklozenge c)$;

VLG2 $(\forall a \in G)(i \blacklozenge a = a \wedge a \blacklozenge fi = fa)$; и

VLG3 $(\forall a \in G)({}^{\leftarrow\square}a \blacklozenge a = i \wedge a \blacklozenge a^{\square\rightarrow} = i \wedge {}^{\leftarrow\blacksquare}a \blacklozenge a = j \wedge$
$\wedge\, a \blacklozenge a^{\blacksquare\rightarrow} = j)$.

И обратно.[1]

ТЕОРЕМА 8´. $(G,\blacklozenge,f)$ явлется L-группой тогда и только тогда, когда имеет место LG1 и $(G,\blacklozenge)$ явлается квазигруппой.

Пусть $(G,\blacklozenge,f)$ L-группа. Пусть, далее, имеет место:

---

[1] Ввиду теоремы 5´, объект $(G,\blacklozenge,f,\leftarrow\square,\square\rightarrow,\leftarrow\blacksquare,\blacksquare\rightarrow,i)$ удовлетворяющий условиям VLG1-VLG3,также, позволим себе назвать L-группой.

(и) $$a * b \overset{\text{деф}}{=} fa \blacklozenge b$$ [1]

для любых $a, b \in G$. Отсюда, ввиду определения 1_2 ($f \in G!$) и теоремы $8'$, находим, что $(G,*)$ квазигруппа. Далее, учитывая (и) и LG1, находим, что $(G,*)$ является и полугруппой. Таким образом, ввиду (и)[1] и определения изотопии[3], мы доказали, что имеет место следующее утверждение:

ТЕОРЕМА 28. Пусть $(G,\blacklozenge,f)$ L-группа. Тогда квазигруппа $(G,\blacklozenge)$ изотопна некоторой группе $(G,*)$.

Подобным способом доказивается следующее утверждение:

УТВЕРЖДЕНИЕ 29. Пусть $(G,*)$ группа, а f подстановка множества G удовлетворяющая условию (о)[2]. Пусть, далее, имеет место

$$a \blacklozenge b \overset{\text{деф}}{=} fa * b$$

для любых $a, b \in G$. Тогда $(G,\blacklozenge,f)$ является L-группой.

ТЕОРЕМА 30. Если (G,Δ,f) является R-группой и L-группой, то (G,Δ) является группой.

Доказательство.

Учитывая определение 1_1 и определение 1_2, находим, что (G,Δ) обладает левой единицей и правой единицей, и поэтому, единицей. Таким образом, ввиду теоремы $8'$, (G,Δ) лупа[3]. Далее, так как, ввиду теоремы 28, лупа (G,Δ) изотопна некоторой группе $(G,*)$, то, на основании теоремы Алберта[3], (G,Δ) является группой.

Теорема доказана.

На табл. 2_1 изображена циклическая группа $(\{1,2,3,4\},*)$. На табл. 2_2 и табл. 2_3 изображены квазигруппы $(\{1,2,3,4\},\square)$ и $(\{1,2,3,4\},\blacklozenge)$, где

$$x \square y \overset{\text{деф}}{=} x * fy,$$

---

[1] ввиду (о), имеет место: $a * b = fa \blacklozenge b \Leftrightarrow fa * b = a \blacklozenge b$.

[2] см. определение 1_2 (определение 1_1).

[3] например: [3-5].

$$x \diamond y \overset{\text{деф}}{=\!=} fx * y, \quad \text{и}$$

$$f \overset{\text{деф}}{=\!=} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}^{1)}.$$

Так как f удовлетворяет условию (o)2), то, ввиду утверждения 11, ({1,2,3,4},□,f) является R-группой, а, ввиду утверждения 29, ({1,2,3,4},♦,f) является L-группой. Отсюда, так как □ = ♦, на основании теоремы 30, находим, что ({1,2,3,4},□) = ({1,2,3,4},♦) является группой. Притом, f ≠ I! 3)

*	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

Табл. 2_1

□	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	1	2
4	3	4	2	1

Табл. 2_2

♦	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	1	2
4	3	4	2	1

Табл. 2_3

Примечание 5.

({1,...,6},□,f), где квазигруппа ({1,...,6},□) изображена на табл. 1_8 и имеет место

$$f \overset{\text{деф}}{=\!=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix},$$

является R-группой. Притом, ({1,...,6},□) является группой! Но ({1,...,6},□,f) не является и L-группой. В самом деле, имеет место:

$$f(3 \,\square\, f4) \,\square\, 5 = 6 \quad \text{и} \quad 3 \,\square\, (4 \,\square\, 5) = 2.$$

Пусть (Q,A) квазигруппа (где $A : Q^2 \rightarrow Q$). Парастрофные операции операции A (^{-1}A, A^{-1}, $\overset{*}{A}$, $(\overset{*}{A}{}^{-1})$ и $({}^{-1}\overset{*}{A})$ определени следующим образом4):

$${}^{-1}A(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} A(z,y) = x;$$

---

1) $f \neq I$.

2) $f \bullet f = I$.

3) см. и примечани 1 и 1´.

4) например: [3-5].

$$A^{-1}(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} A(x,z) = y;$$

$$\overset{*}{A}(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} A(y,x) = z;$$

$$(\overset{*}{A}{}^{-1})(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} A(y,z) = x^{1)}; \quad \text{и}$$

$$({}^{-1}\overset{*}{A})(x,y) = z \overset{\text{деф}}{\Longleftrightarrow} A(z,x) = y^{2)}.$$

Если (Q,A) группа и f унарная операция - взятие обратного элемента, то имеет место:

$$ {}^{-1}A(a,b) = A(a,fb) \quad \text{и}$$

$$A^{-1}(a,b) = A(fa,b).$$

Отсюда, ввиду утверждения 11 и утверждения 29, в том же порядке, находим, что имеет место:

$(Q, {}^{-1}A, f)$ R-группа; и

(Q, A^{-1}, f) L-группа.

Отсюда, ввиду теоремы 27, находим, что имеет место:

$(Q, ({}^{-1}\overset{*}{A}), f)$ L-группа; и

$(Q, (\overset{*}{A}{}^{-1}), f)$ R-группа.

Мы доказали следующее утверждение:

ТЕОРЕМА 31. Пусть (Q,A) группа и f унарная операция - взятие обратного элемента. Тогда

а) $(Q, {}^{-1}A, f)$ и $(Q, (\overset{*}{A}{}^{-1}), f)$ R-группы; и

б) (Q, A^{-1}, f) и $(Q, ({}^{-1}\overset{*}{A}), f)$ L-группы.3)

---

1) притом, имеет место: $(\overset{*}{A}{}^{-1}) = {}^{-1}(\overset{*}{A})$.

2) притом, имеет место: $({}^{-1}\overset{*}{A}) = (\overset{*}{A})^{-1}$.

3) притом, $(Q,\overset{*}{A})$ группа.

Примечание 6.

Правое и левое деление в квазигруппе (Q,A), в том же порядке, определени следующим образом:

$$x \backslash y = z \overset{\text{деф}}{\Longleftrightarrow} A(x,z) = y; \quad \text{и}$$

$$x / y = z \overset{\text{деф}}{\Longleftrightarrow} A(z,x) = y.$$

Таким образом, ввиду определений парастрофов квазигруппы (Q,A), находим, что имеет место:

$$\backslash = A^{-1} \quad \text{и} \quad / = ({}^{-1}\overset{*}{A}).$$

Отсюда, ввиду теоремы 31, находим, что имеет место: если (Q,A) группа и f унарная операция - взятие обратного элемента, то $(Q,\backslash,f)$ и $(Q,/,f)$ являются L-группамы.

Примечание 7.

В [6] В. Дэвидэ-ом введени и рассматривани группоиды и квазигруппы $(Q,\cdot)$ удовлетворяющие условию

$$(\forall a \in Q)(\forall b \in Q)(\forall c \in Q)(a \cdot b) \cdot c = f_1 a \cdot (f_2 b \cdot f_3 c),$$

где f_1, f_2, f_3 унарные операции в Q. В частности рассматриван случай когда $(Q,\cdot)$ квазигруппа а f_1, f_2 и f_3 ее автоморфизмы. Эту структуру автор называет А-группа. Таким образом, клики являются и специальными А-группамы.

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REZIME

O DVE KLASE ALGEBRI BLISKIH GRUPAMA

U radu se uvode i razmatraju dve klase algebri među kojima se pored grupa nalaze, na primer, klike [1] i poluklike [2].

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THIS ISSUE IS DEDICATED TO

Academician Bogoljub Stanković

on the occasion of his 65th birthday and 40 years of scientific work

Bogoljub Stanković was born on November 1, 1924, in Botoš, Vojvodina. He graduated in mathematics from the University of Belgrade in 1949, the year of the beginning of his scientific work in mathematics. He received his Ph. D. from the Serbian Academy of Science and Arts in 1954. His thesis already contained profound results on singular integral equations. In the same year, Dr. Stanković became one of the founders of the Department of Mathematics at the University of Novi Sad. Dr. Bogoljub Stanković is Full Professor since 1964.

Working on his doctoral thesis, he became interested in generalized functions, quite a new field in early fifties. These were the years when Laurent Schwartz and Jan Mikusinski were formulating the theories known today as Schwartz's theory of distributions and Mikusinski's operator calculus. Their research influenced many mathematicians throughout the world to work on their further development and application in different fields. Prof. Stanković worked on applications of Mikusinski operators to different classes of ODEs and PDEs in the sixties and seventies, and his papers published in leading mathematics journals, were highly valued by both Yugoslav and foreign specialists in the field. In 1962 Prof. Stanković wrote a monograph on the Mikusinski calculus which is up to now the only such book in Serbocroatian. In the late seventies he started to work intensively on distribution theory, in particular on the asymptotic behaviour of distributions and Abelian and Tauberian theorems for integral transforms. This work resulted in several papers and two monographs (as a co-author), the last published in 1989 by Teubner Verlag in Leipzig. Altogether, Prof. Stanković has written more than a 100 mathematics papers and 9 textbooks and monographs. He was referee for several journals, editor of the Proceedings of the Conference on Generalized Functions and Convergences held in Dubrovnik in 1987 (and also its main organizer), published by Plenum Publishing Company, and a reviewer for Zentralblatt für Mathematik und Ihre Grenzgebiete.

Prof. Stanković has contributed immensly to the teaching of mathematics at the University of Novi Sad. He founded several courses at undergraduate and graduate levels, mostly in functional and complex analysis. He is well known both among his colleagues and his students as an excellent teacher. Parallel to his scientific work, he established The Generalized Function Group, which has been working for more than thirty years. In particular, this group was an excellent beginning for several of his pupils, some of who are today well known Yugoslav mathematicians. He has organized joint work with several mathematics institutes and departments, either through agreements between the Academy of Sciences or between the institutions themselves. At the present, Prof. Stanković is the principal investigator for two scientific projects (one supported by the American NSF). It should be noted that he is not only the head of the Group and these projects, but also one of their most active members.

Prof. Stanković has also contributed greatly to the University. He was the Director of the Institute of Mathematics, Dean of the Faculty of Natural Sciences and Mathematics, and Rector of the University of Novi Sad. In 1963 he was elected to the Serbian Academy of Sciences and Arts, and in 1979 he became the first president of the Academy of Sciences and Arts of Vojvodina for a period of four years.

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p(3,-1) - FINSLER STRUCTURES AND THEIR LIFTS

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Abstract

On a differentiable manifold M and on the total space E of a vector bundle there are defined $p(3,-1)$, $P(3,-1)$ - Finsler structures respectively and the Finsler connections which are compatible with these structures are determined. Furthermore the invariants of the Finsler connection transformations group are established, the integrability conditions are determined and some special cases are given.

0. Introduction

The main purpose of the present paper is to introduce the notion of $p(3,-1)$-Finsler structure on an n-dimensional C^∞- manifold and the notion of $P(3,-1)$-structure on the total space E of a vector bundle $\xi = (E,\pi,M)$ and to study these structures by the method used by R. Miron [14] and by the first author in [1].

A $P(3,-1)$-structure on E we simply call a *Walker structure.* It is similar to the Kentaro Yano structure on E, [1].

In §1 we recall the notion of a Finsler connection on M [11], [12] and the notion of a distinguished connection on E [15], [16], [17]. We introduce, in §2, the notion of $p(3,-1)$-Finsler structure on M and all $p(3,-1)$-Finsler connections are determined in §3, using Obata's operators (2.5). In §4, we consider the group G_p of transformations of $p(3,-1)$-Finsler connections and prove that it has twenty invariants, which

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are Finsler tensor fields. By the lift in Miron's sense [14], on *TM* of one $p(3,-1)$-Finsler structure from *M*, in §5, we obtain three structures on *TM*, that each of them, taken adequately on the horizontal distribution *HTM* and on the vertical distribution *VTM* of $T(TM)$, are $p(3,-1)$-Finsler structures. By means of the invariants of §3, in §6, we give characterizations of the integrability of type I, II or III for the $p(3,-1)$-Finsler structures based only on Finsler geometry. Finally, in §6, we define and treat the (h,v) $P(3,-1)$-structure on the total space *E* of a vector bundle ξ, which we shall call a *(h,v)-Walker structure* on *E*.

In particular, we have on *M* an almost product Finsler structure [3], [8], [9], [10], or a (p,ξ,η)-Finsler structure [6].

The terminology and notations are all according to M. Matsumoto [11] and R. Miron [12, 13, 14, 15, 16, 17].

1. Preliminaries

Let *M* be a differentiable C^∞ - manifold, paracompact, of dimension n, let $T(M) = (TM,\pi,M)$ be its tangent bundle and let N be a non-linear connection on TM: $TM = N + (TM)^V$. We denote by $(x^i, y^i)(i,j,k,l=1,2,\dots,n)$ the canonical coordinates on *TM*. Then $\delta_i = \partial_i - N^k_i \dot{\partial}_k$ $(\partial_i = \partial/\partial x^i,\ \dot{\partial}_i = \partial/\partial y^i,\ \delta_i = \delta/\delta x^i)$ is a local basis of the horizontal distribution *N*, and $\dot{\partial}_i$ is a local basis of the vertical distribution $(TM)^V$. The dual basis is $(dx^i, \delta y^i)$, where $\delta y^i = dy^i + N^i_k dx^k$. We have:

$$(1.1)\qquad [\delta_j, \delta_k] = R^i_{\ jk}\dot{\partial}_i,\quad [\delta_j, \dot{\partial}_k] - [\delta_k, \dot{\partial}_j] = t^i_{\ jk}\dot{\partial}_i,\quad [\dot{\partial}_j, \dot{\partial}_k] = 0$$

We denote by

$$(1.2)\qquad R^i_{\ jk} = \delta_k N^i_j - \delta_j N^i_k,\qquad t^i_{\ jk} = \dot{\partial}_k N^i_j - \dot{\partial}_j N^i_k$$

the curvature and torsion tensor fields of *N*.

A Finsler connection on *M* is a triad $FT = (N,F,C)(=\nabla)$, where *N* is a non-linear connection on *TM*, and *F*, respectively, *C* are the *h*-and *v*-connection coefficients given by:

$$(1.3)\qquad \nabla_{\delta_k}\delta_j = F^i_{\ jk}\delta_i,\quad \nabla_{\delta_k}\dot{\partial}_j = F^i_{\ jk}\dot{\partial}_i,\quad \nabla_{\dot{\partial}_k}\delta_i = C^i_{\ jk}\delta_i,\quad \nabla_{\dot{\partial}_k}\dot{\partial}_j = C^i_{\ jk}\dot{\partial}_i$$

With $|$ and $\Big|$ we denote the *h*-and *v*-covariant derivatives with respect to *FT*. For example, for a Finsler tensor field of the type (1,1) with the components $K^i_j(x,y)$ we have

$$(1.4)\qquad K^i_{j|k} = \delta_k K^i_j + F^i_{sk}K^s_j - F^s_{jk}K^s_j,\quad K^i_j\Big|_k = \dot{\partial}_k K^i_j + C^i_{sk}K^s_j - C^s_{jk}K^i_s .$$

The torsion Finsler tensor filed of $F\Gamma$ will be denoted by: T^i_{jk}, R^i_{jk}, C^i_{jk}, P^i_{jk}, S^i_{jk}, and the curvature Finsler tensor fields of $F\Gamma$ will be denoted by: $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$, $S_j{}^i{}_{kh}$, [11], [12].

If $F\Gamma = (N,F,C)$ and $F\bar{\Gamma} = (\bar{N},\bar{F},\bar{C})$ are two Finsler connections on M, then a unique triad of Finsler tensor fileds (A,B,D) is determined such that:

$$(1.5)\qquad \bar{N}^i_j = N^i_j - A^i_j;\quad \bar{F}^i_{jk} = F^i_{jk} + C^i_{js}A^s_k - B^i_{jk},\quad \bar{C}^i_{jk} = C^i_{jk} - D^i_{jk} .$$

Conversely: given the Finsler connection $F\Gamma = (N,F,C)$ and the triad (A,B,D) of Finsler tensor fields, the connection $F\bar{\Gamma} = (\bar{N},\bar{F},\bar{C})$ given by (1.5) is a Finsler connection. The map $F\Gamma \to F\bar{\Gamma}$ defined by (1.5) is called a transformation of Finsler connections.

If $F\mathring{\Gamma} = (\mathring{N},\mathring{F},\mathring{C})$ is a connection fixed on M, then the h - and v - covariant derivatives with respect to $F\mathring{\Gamma}$ will be denoted, resceptively, by $\overset{\circ}{|}$ and $\overset{\circ}{\Big|}$.

Because a linear connection $D = (\Gamma^i_{jk})$ on TM, expressed in the adapted basis $(\delta_i,\dot{\partial}_i)$ of a non-linear connection N and the vertical distribution $(TM)^v$ has eight coefficients, in [15], [16] was introduced the notion of a d-connection on the total space E of a vector bundle $\xi = (E,\pi,M)$ (in particularly on $E = TM$), that is, fundametal in the geometry of E. We shall recall in short this important notion.

Let $\xi = (E,\pi,M)$ be a vector bundle of the class C^∞. We suppose that the total space E has $n+m$ dimensions, the base M has n dimensions and the local fibre $E_x = \pi^{-1}(x)$, $x \in M$, is a real vector space of dimensions m. $\mathcal{X}(E)$, $\mathcal{X}^*(E)$, $\sigma^p_q(E)$ are well-known sections.

We denote by N a non-linear connection (horizontal distribution) on E and by V the vertical distribution complementary with N:

$$T_uE = H_uE + V_uE,\quad \forall u \in E.$$

A linear connection D on E is called a *distinguished connection* (for short a d-connection) if: $D_ZX \in HE$, $D_ZY \in VE$, for $\forall X \in HE$, $\forall Y \in VE$, $\forall Z \in \mathcal{X}(E)$.

It follows that for a *d*-connection on *E* we have a unique decomposition: $D = D^H + D^V$. D^H is the *h*-covariant derivatives and D^V is *v*-covariant derivative of *D*.

In the canonical coordinates (x^i, y^a) of the point $u = (u^\alpha) \in E$, $i=\overline{1,n}$, $a=\overline{1,m}$, $\alpha=\overline{1,n+m}$, we have $(\delta_i, \dot\partial_a)$, $(dx^i, \delta y^a)$ the dual frames adapted to $N\ (N^a_i(x,y))$.

The local components $(L^i_{jk}(x,y),\ L^a_{bk}(x,y),\ C^i_{jc}(x,y),\ C^a_{bc}(x,y))$ of a *d*-connection on *E* are given by:

$$(1.6)\qquad D_{\delta_k}\delta_j = L^i_{jk}\delta_i,\quad D_{\delta_k}\dot\partial_b = L^a_{bk}\dot\partial_a,\quad D_{\dot\partial_c}\delta_j = C^i_{jc}\delta_i,\quad D_{\dot\partial_c}\dot\partial_b = C^a_{bc}\dot\partial_a .$$

We denote by $X^H (X^V)$ and $\omega^H (\omega^V)$, the horizontal (vertical) components of $X \in \mathfrak{X}(E)$ and respectively $\omega \in \mathfrak{X}^*(E)$.

A tensor field *t* on *E* is called a *distinguished tensor field (d-tensor filed)* of the type $\begin{bmatrix} pr \\ qs \end{bmatrix}$ if it has the property:

$$(1.7)\qquad t\left[\underset{1}{\omega},\ldots,\underset{p}{\omega};\ \underset{1}{X},\ldots,\underset{q}{X};\ \underset{p+1}{\omega},\ldots,\underset{p+r}{\omega};\ \underset{q+1}{X},\ldots,\underset{q+s}{X}\right] =$$

$$= t\left[\underset{1}{\omega}^H,\ldots,\underset{p}{\omega}^H;\ \underset{1}{X}^H,\ldots,\underset{q}{X}^H;\ \underset{p+1}{\omega}^V,\ldots,\underset{p+r}{\omega}^V;\ \underset{q+1}{X}^V,\ldots,\underset{q+s}{X}^V\right],$$

$$\forall \underset{\alpha}{X} \in \mathfrak{X}(E),\ \forall \underset{\alpha}{\omega} \in \mathfrak{X}^*(E).$$

Proposition 1.1. *If t is a tensor field on E of the type (p,q), then it determines* 2^{p+q} *d-tensor fileds on E of the type* $\begin{bmatrix} p-r & r \\ q-s & s \end{bmatrix}$ $(r=0,1,\ldots,p;$ $s=0,1,\ldots q)$.

It follows that the torsion tensor field *T* of the *d*-connection $D\Gamma$, (1.6), is characterized by five *d*-tensor fields of components: T^i_{jk}, R^a_{jk}, C^i_{jc}, P^a_{jc}, S^a_{bc} $(T(X,Y) = -T(Y,X))$, where:

$$(1.8)\qquad \begin{cases} T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad R^a_{jk} = \delta_k N^a_j - \delta_j N^a_k, \quad C^i_{jc}, \\ P^a_{jc} = \dot\partial_c N^a_j - L^a_{cj}, \quad S^a_{bc} = C^a_{bc} - C^a_{cb}. \end{cases}$$

Also, the curvature tensor filed *R* of the *d*-connection $D\Gamma$, (1.6), is characterized by six *d*-tensor fields of components: $R_j{}^i{}_{kl}$, $R_b{}^a{}_{kl}$, $P_j{}^i{}_{kl}$, $P_b{}^a{}_{kd}$, $S_j{}^i{}_{cd}$, $S_b{}^a{}_{cd}$ $((R(X,Y)Z^H)^V = 0,\ \ (R(X,Y)Z^V)^H = 0)$, where:

$$(1.9)_1 \quad \begin{cases} R_j{}^i{}_{kl} = \delta_l L^i_{jk} - \delta_k L^i_{jl} + L^h_{jk} L^i_{hl} - L^h_{jl} L^i_{hk} + C^i_{jc} R^c_{kl}\,, \\ R_b{}^a{}_{kl} = \delta_l L^a_{bk} - \delta_k L^a_{bl} + L^f_{bk} L^a_{fl} - L^f_{bl} L^a_{fk} + C^a_{bc} R^c_{kl}\,, \end{cases}$$

$$(1.9)_2 \quad \begin{cases} P_j{}^i{}_{kd} = \dot{\partial}_d L^i_{jk} - C^i_{jd|k} + C^i_{jf} P^f_{kd} \\ P_b{}^a{}_{kd} = \dot{\partial}_d L^a_{bk} - C^a_{bd|k} + C^a_{bf} P^f_{kd}\,, \end{cases}$$

$$(1.9)_3 \quad \begin{cases} S_j{}^i{}_{cd} = \dot{\partial}_d C^i_{jc} - \dot{\partial}_c C^i_{jd} + C^h_{jc} C^i_{hd} - C^h_{jd} C^i_{hc}\,, \\ S_b{}^a{}_{cd} = \dot{\partial}_d C^a_{bc} - \dot{\partial}_c C^a_{bd} + C^f_{bc} C^a_{fd} - C^f_{bd} C^a_{fc}\,. \end{cases}$$

If $E = TM$, we have $n=m$ and we can remain in the present notations on E, with the convention to assimilate x^a with $\delta^a_i x^i$.

Then, on TM have in general:

$$(1.10) \qquad L^i_{jk} \neq \delta^i_a \delta^b_j L^a_{bk}\,, \qquad C^i_{jc} \neq \delta^i_a \delta^b_j C^a_{bc}. \quad ;$$

A d-connection $D\Gamma$ on TM for which (1.10) is tranformed into equalities is called a *d-connection of the Finsler type* on TM (in [23] such a d-connection is called a d-normal connection).

Remark 1.1. A d-tensor filed on the manifold base M is also called a Finsler tensor filed on M [11], [12].

2. $p(3,-1)$-Finsler structures

Let M be an n-dimensional differentiable manifold of class C^∞ and $x=(x^i)$ and $y=(y^i)$ denote a point of M and a supporting element, respectively.

Definition 2.1. *A Finsler tensor field $p(x,y) \neq 0$ of type (1,1) and of class C^∞ is called a $p(3,-1)$-Finsler structures of index k, it is satisfies*

$$(2.1) \qquad p^3 - p = 0, \quad rank\, \| p(x,y) \| = n - k\,, \quad \forall (x,y) \in TM\,,$$

where k is integer and $0 \le k < n$.

We denote by $\mathcal{H}$ and V complementary distributions of tangent bundle $T(M)$, corresponding to the projection operators $h(x,y)$ and $v(x,y)$ given by

$$(2.2)\qquad h = p^2,\quad v = -p^2 + I \qquad (I \text{ - the identity operator}).$$

We have $\dim \mathcal{H}_{(x,y)} = n - k$, $\dim V_{(x,y)} = k$ and

$$(2.3)\qquad \begin{cases} ph = hp = p & pv = vp = o \\ p^2 h = h & p^2 v = o \end{cases}$$

that is, p acts on $\mathcal{H}$ as an almost product operator and on V as a null operator.

Remark 2.1. If the rank of $p(x,y)$ is n, then $h = I$ and $v = 0$ and $p(x,y)$ satisfies: $p^2 = I$. Consequently the $p(3,-1)$-Finsler structure of *minimum index* is an almost product Finsler structure [3], [8], [9]; the dimension n not must be even.

Remark 2.2. If the $p(3,-1)$-Finsler structure is of *index 1*, then $\mathcal{H}$ is $(n-1)$-dimensional and V is one-dimensional. Consequently if we denote by $p^i_j(x,y)$, $h^i_j(x,y)$ and $v^i_j(x,y)$ $(i, j, \ldots, = 1, 2, \ldots, n)$ the local components of $p(x,y)$, $h(x,y)$ and $v(x,y)$ respecitvely, then $v^i_j(x,y)$ should have the form $v^i_j = \eta_j \xi^i$, where $\eta(x,y)$ and $\xi(x,y)$ are covariant contravariant Finsler vector fields respectively. Taking into account the relations (2.2) and (2.3) we have

$$(2.4)\qquad p^i_r p^r_j = \delta^i_j - \eta_j \xi^i,\quad p^i_j \xi^j = o,\quad \eta_i p^i_j = o,\quad \eta_i \xi^i = 1 .$$

Thus a $p(3,-1)$-Finsler structure of index 1 is equivalent to a (p,ξ,η)-Finsler structure (see [6], p.26, Definition 3.2 and Remark 3.2).

Definition 2.2. *We shall call Obata's operators of the $p(3,-1)$-Finsler structure,* (2.1), *the Finsler tensor fields $\underset{1}{O}{}^{rs}_{ij}$ and $\underset{2}{O}{}^{rs}_{ij}$ given by*

$$(2.5)\qquad \begin{cases} \underset{1}{O}{}^{rs}_{ij} = \dfrac{1}{2}\left[\delta^r_i \delta^s_j - \delta^r_i v^s_j - v^r_i \delta^s_j + p^r_i p^s_j + 3\, v^r_i v^s_j\right], \\[2ex] \underset{2}{O}{}^{rs}_{ij} = \dfrac{1}{2}\left[\delta^r_i \delta^s_j + \delta^r_i v^s_j + v^r_i \delta^s_j - p^r_i p^s_j - 3\, v^r_i v^s_j\right], \end{cases}$$

These operators have the symmetry $\underset{\alpha}{O}{}^{rs}_{ij} = \underset{\alpha}{O}{}^{sr}_{ji}$ ($\alpha=1,2$) and act on a Finsler tensor field K of type (1,2) as $(\underset{\alpha}{O}K)^i_{jk} = \underset{\alpha}{O}{}^{ri}_{js} K^s_{rk}$. The product $\underset{\alpha}{O}\,\underset{\beta}{O}$ is defined by $(\underset{\alpha}{O}\,\underset{\beta}{O})^{ti}_{jm} = \underset{\alpha}{O}{}^{ri}_{js}\,\underset{\beta}{O}{}^{ts}_{rm}$ ($\alpha,\beta = 1,2$).

Proposition 2.1. $\underset{1}{O},\underset{2}{O}$ *are the supplementary projectors on the module* τ^1_2 *of the tensor fields of type* (1,2):

$$(2.6) \qquad \underset{1}{O} + \underset{2}{O} = I, \quad \underset{\alpha}{O}^2 = \underset{\alpha}{O}, \quad \underset{1}{O}\,\underset{2}{O} = \underset{2}{O}\,\underset{1}{O} = o \quad (\alpha=1,2),$$

where I *is the identity given by* $\delta^r_j\,\delta^i_s$: $IK = K$.

Proposition 2.2. $\underset{2}{O}X = o$ *(resp.* $\underset{1}{O}X = o$*) has solutions, and its general solutions are given by* $X = \underset{1}{O}Y$ *(resp.* $X = \underset{2}{O}Y$*), where* $Y \in \tau^1_2$ *is arbitrary.*

3. $p(3,-1)$-Finsler connections

An important problem concerning a $p(3,1)$-Finsler connections on M is to determine the existence and arbitrariness of Finsler connections with respect to wich $p^i_j(x,y)$ is covariantly constant.

Definition 3.1. *Let* $p^i_j(x,y)$ *be a* $p(3,1)$*-Finsler structure of index* k. *A Finsler connection* $F\Gamma = (N,F,C)$ *is called a* $p(3,1)$*-Finsler connection or compatible with* p^i_j (2.1), *if* p^i_j *is covariantly constant:*

$$(3.1) \qquad p^i_{j|k} = 0 \qquad p^i_j\Big|_k = 0$$

Proposition 3.1. *With respect to a Finsler connection* $F\Gamma$ *compatible with a* $p(3,1)$*-Finsler structure* $p^i_j(x,y)$, (2.1), *the tensor fields* $h^i_j(x,y)$ and $v^i_j(x,y)$ *are covariantly cosntant:*

$$(3.2) \qquad h^i_{j|k} = o, \quad h^i_j\Big|_k = o, \quad v^i_{j|k} = o, \quad v^i_j\Big|_k = o\,.$$

Theorem 3.1. *(a) Obata tensor fields* $\underset{1}{O}$ *and* $\underset{2}{O}$ *are covariantly constant with respect to any* $p(3,1)$*-Finsler connections* $F\Gamma$; *(b) The Finsler tensor fields* $\underset{2}{O}{}^{ir}_{sj}R_r{}^s{}_{kl}$, $\underset{2}{O}{}^{ir}_{sj}P_r{}^s{}_{kl}$, $\underset{2}{O}{}^{ir}_{sj}S_r{}^s{}_{kl}$ and *h-and v-covariant derivatives of every order vanish for every* $F\Gamma$ *with the property* (3.1).

Proof. The property (a) results immediately from (2.5), (3.1) and (3.2). By the Ricci identities applying to p^i_j and taking into account (a) we get the statement (b).

Theorem 3.2. *Let* $F\mathring{\Gamma} = (\mathring{N}, \mathring{F}, \mathring{C})$ *be a fixed Finsler connection on* M. *Then there exist* $p(3,1)$*-Finsler connections with respect to the* $p(3,-1)$*-Finsler structures* (2.1); *one of these is:*

$$(3.3)\quad \begin{cases} \check{N}^i_j = \mathring{N}^i_j, \quad \check{F}^i_{jk} = \mathring{F}^i_{jk} + \frac{1}{2}\left\{p^i_r p^r_j \overset{\circ}{|}_k - v^i_j \overset{\circ}{|}_k + 3v^i_r v^r_j \overset{\circ}{|}_k\right\} \\ \check{C}^i_{jk} = \mathring{C}^i_{jk} + \frac{1}{2}\left\{p^i_r p^r_j \overset{\circ}{|}_k - v^i_j \overset{\circ}{|}_k + 3v^i_r v^r_j \overset{\circ}{|}_k\right\} \end{cases}$$

Proof. By straightforward calculus, (3.3) satisfies (3.1).

Now, we determine all the $p(3,-1)$-Finsler connections based on Proposition 2.2. Let $F\dot{\Gamma}$ be a fixed Finsler connection. Then any Finsler connection $F\Gamma$ on M can be expressed in the form

$$(3.4)\quad N^i_j = \dot{N}^i_j - A^i_j, \quad F^i_{jk} = \dot{F}^i_{jk} + \dot{C}^i_{jk} A^r_k - B^i_{jk}, \quad C^i_{jk} = \dot{C}^i_{jk} - D^i_{jk},$$

where A, B, D are arbitrary Finsler tensor fields [12], [13].

We put $F\dot{\Gamma} = F\check{\Gamma}$ in (3.4), where $F\check{\Gamma} = (\check{N}, \check{F}, \check{C})$ is given by (3.3).

In order that $F\Gamma$ is a $p(3,-1)$-Finsler connection, that is, the equation (3.1) hold for $F\Gamma$ given by (3.4), it is necessary and sufficient that A, B, D satisfy

$$B^i_{jk} - (p^i_s p^h_j + v^i_s v^h_j)B^s_{hk} = 0, \qquad D^i_{jk} - (p^i_s p^h_j + v^i_j v^h_j)D^s_{hk} = 0,$$

which, after one long calculus (analogous to [6], p.23-24), is equivalent to

$$\underset{2}{O} B = 0 \qquad \underset{2}{O} D = 0 .$$

From Proposition 2.2, however, the last system has solutions in B^i_{jk}, D^i_{jk} for any Finsler tensor field $A^i_j = X^i_j$. Thus, we have

Theorem 3.3. *Let* $F\mathring{\Gamma}$ *be a fixed Finsler connection. The set of all* $p(3,-1)$*-Finsler connection* $F\Gamma$ *with respect to the* $p(3,-1)$*-Finsler structure* (2.1) *is given by*

$$(3.5)\quad \begin{cases} N^i_j = \overset{\circ}{N}{}^i_j - X^i_j\,, \\ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jr}X^r_k + \dfrac{1}{2}\Big\{p^i_s\Big[p^s_{j\overset{\circ}{|}k} + p^s_j\overset{\circ}{|}_m X^m_k\Big] - \Big[v^i_{j\overset{\circ}{|}k} + v^i_j\overset{\circ}{|}_m X^m_k\Big] + \\ \qquad 3v^i_s\Big[v^s_{j\overset{\circ}{|}k} + v^s_j\overset{\circ}{|}_m X^m_k\Big]\Big\} + \underset{1}{O}{}^{is}_{mj}\,Y^m_{sk}\,, \\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \dfrac{1}{2}\Big\{p^i_s p^s_j\overset{\circ}{|}_k + v^i_j\overset{\circ}{|}_k + 3v^i_s v^s_j\overset{\circ}{|}_k\Big\} + \underset{1}{O}{}^{is}_{mj}\,Z^m_{sk}\,, \end{cases}$$

where X^i_j, Y^i_{jk}, Z^i_{jk} *are arbitrary Finsler tensor fields.*

Remark 3.1. The $p(3,-1)$-Finsler connection $F\check{\Gamma} = (\check{N}, \check{F}, \check{C})$ given by (3.3) is obtained from (3.5) for $X^i_j = 0$, $Y^i_{jk} = 0$, $Z^i_{jk} = 0$.

Corrolary 3.1. *If $F\overset{\circ}{\Gamma}$ is a fixed p(3,-1)-Finsler connection, then the set of all p(3,-1)-Finsler connections $F\Gamma$ is given by:*

$$(3.6)\quad N^i_j = \overset{\circ}{N}{}^i_j - X^i_j\,,\quad F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jr}X^r_k + \underset{1}{O}{}^{ir}_{sj}\,Y^s_{rk}\,,\quad C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \underset{1}{O}{}^{ir}_{sj}\,Z^s_{rk}\,,$$

where X^i_j, Y^i_{jk}, Z^i_{jk} *are arbitrary Finsler tensor fields.*

We denote by $F\Gamma(\overset{\circ}{N})$ the Finsler connections having the same non-linear connection N.

Theorem 3.4. *The set of all p(3,-1)-Finsler connections $F\Gamma(\overset{\circ}{N})$ is given by*

$$(3.7)\quad \begin{aligned} F^i_{jk} &= \overset{\circ}{F}{}^i_{jk} + \frac{1}{2}\Big\{p^i_s\,p^s_{j\overset{\circ}{|}k} - v^i_{j\overset{\circ}{|}k} + 3v^i_s v^s_{j\overset{\circ}{|}k}\Big\} + \underset{1}{O}{}^{is}_{jr}\,Y^r_{sk}\,, \\ C^i_{jk} &= C^i_{jk} + \frac{1}{2}\Big\{p^i_s\,p^s_j\overset{\circ}{|}_k - v^i_j\overset{\circ}{|}_k + 3v^i_s v^s_j\overset{\circ}{|}_k\Big\} + \underset{1}{O}{}^{is}_{jr}\,Z^r_{sk}\,, \end{aligned}$$

where $F\overset{\circ}{\Gamma}$ is a fixed Finsler connection and Y^i_{jk}, Z^i_{jk} are arbitrary Finsler tensor fields.

4. The group of transformations of $p(3,-1)$-Finsler connections

Let us consider the transformations $F\Gamma(N) \to F\bar{\Gamma}(N)$, [13], of p(3,-1)-Finsler connections, which preserve the non-linear connection N. Owing to Theorem 3.4. they are given by

$$\bar{N}^i_j = N^i_j\,,\quad \bar{F}^i_{jk} = F^i_{jk} + \underset{1}{O}{}^{ir}_{sj}\,Y^s_{rk}\,,\quad \bar{C}^i_{jk} = C^i_{jk} + \underset{1}{O}{}^{ir}_{sj}\,Z^s_{rk}\,.$$

Theorem 4.1. *The set of all transformations* (4.1), *with the mapping product as a law of composition, form an Abelian group* G_p, *which is isomorphic with the additive group of pair of Finsler tensor fields* $\left[\underset{1}{O}{}^{ir}_{sj} Y^s_{rk}, \underset{1}{O}{}^{ir}_{sj} Z^s_{rk}\right]$.

By a straightforward calculus we can prove

Theorem 4.2. *The following Finsler tensor fields are invariants by the action of the group* G_p:

$$R^i_{jk}\ ,\quad t^i_{jk}\ , \tag{4.2}$$

$$\overset{1}{N}{}^i_{jk} = \underset{2}{O}{}^{ri}_{pq}\underset{1}{O}{}^{ps}_{jk}T^q_{rs}\ ,\quad \tilde{C}^i_{jk} = \underset{2}{O}{}^{ri}_{pq}\underset{1}{O}{}^{ps}_{jk}C^q_{rs}\ ,\quad \tilde{P}^i_{jk} = \underset{2}{O}{}^{ri}_{pq}\underset{1}{O}{}^{ps}_{jk}P^q_{rs}\ ,\quad \overset{2}{N}{}^i_{jk} = \underset{2}{O}{}^{ri}_{pq}\underset{1}{O}{}^{ps}_{jk}S^q_{rs}\ , \tag{4.3}$$

$$\begin{cases} \overset{1}{T}{}^i_{jk} = h^i_m T^m_{jk} + p^r_j p^s_k T^i_{rs} - \left[p^r_j T^m_{rk} + p^s_k T^m_{js}\right] p^i_m\ , \\ \overset{1}{R}{}^i_{jk} = h^i_m R^m_{jk} + p^r_j p^s_k R^i_{rs} - \left[p^r_j R^m_{rk} + p^s_k R^m_{js}\right] p^i_m\ , \\ \overset{1}{C}{}^i_{jk} = h^i_m C^m_{jk} + p^r_j p^s_k C^i_{rs} - \left[p^r_j C^m_{rk} + p^s_k C^m_{js}\right] p^i_m\ , \\ \overset{1}{P}{}^i_{jk} = h^i_m P^m_{jk} + p^r_j p^s_k P^i_{rs} - \left[p^r_j P^m_{rk} + p^s_k P^m_{js}\right] p^i_m\ , \\ \overset{1}{S}{}^i_{jk} = h^i_m S^m_{jk} + p^r_j p^s_k S^i_{rs} - \left[p^r_j S^m_{rk} + p^s_k S^m_{js}\right] p^i_m\ , \end{cases} \tag{4.4}$$

$$\begin{cases} \overset{2}{R}{}^i_{jk} = h^i_m R^m_{jk} + p^r_j p^s_k R^i_{rs} + \left[p^r_j R^m_{rk} + p^s_k R^m_{js}\right] p^i_m\ , \\ \overset{2}{C}{}^i_{jk} = h^i_m C^m_{jk} - p^r_j p^s_k C^i_{rs} - \left[p^r_j C^m_{rk} - p^s_k C^m_{js}\right] p^i_m\ , \\ \overset{2}{P}{}^i_{jk} = h^i_m P^m_{jk} - p^r_j p^s_k P^i_{rs} + \left[p^r_j P^m_{rk} - p^s_k P^m_{js}\right] p^i_m\ , \end{cases} \tag{4.5}$$

$$\begin{cases} \overset{3}{T}{}^i_{jk} = h^i_m T^m_{jk} + \left[p^r_j P^m_{kr} + p^s_k P^m_{js}\right] p^i_m\ , \\ \overset{3}{R}{}^i_{jk} = h^i_m R^m_{jk} + p^r_j p^s_k S^i_{rs} + \left[p^r_j C^m_{kr} + p^s_k C^m_{js}\right] p^i_m\ , \\ \overset{3}{C}{}^i_{jk} = h^i_m C^m_{jk} - p^r_j p^s_k C^i_{sr} - \left[p^r_j S^m_{rk} + p^s_k R^m_{js}\right] p^i_m\ , \\ \overset{3}{P}{}^i_{jk} = h^i_m P^m_{jk} - p^r_j p^s_k P^i_{sr} - p^s_k T^m_{js}\, p^i_m\ , \end{cases} \tag{4.6}$$

$$(4.6)\quad \begin{cases} \overset{3}{S}{}^i_{jk} = h^i_m S^m_{jk} + p^r_j p^s_k R^i_{rs} - \left[p^r_j S^m_{rk} + p^s_k R^m_{js}\right] p^i_m , \\ \tilde{T}^i_{jk} = p^r_j p^s_k T^i_{rs} + \left[p^r_j P^m_{rk} + p^s_k P^m_{sj}\right] p^i_m , \end{cases}$$

Theorem 4.3. *The Finsler tensor fields* $\overset{1}{N}{}^i_{jk}$, $\overset{2}{N}{}^i_{jk}$, $\overset{1}{T}{}^i_{jk}$, $\overset{1}{S}{}^i_{jk}$ *vanish if and only if there exists on* M *an* h *- and* v *- semi-symmetric* $p(3,-1)$*-Finsler connection* $F\Gamma(N)$.

5. $\tilde{P}(3,-1)$-structures on the tangent bundle $T(M)$

The lift of one p(3,-1)-Finsler structure. Let M be a differentiable manifold of the class C^∞, with n dimensions, $T(M) = (TM, \pi, M)$ its tangent bundle and N a fixed non-linear connection on TM.

A $\tilde{P}(3,-1)$-structure of index k' on TM is given by a tensor field $\tilde{P} \in \tau^1_1(TM)$ with the property:

$$(5.1)\qquad \tilde{P}^3 - \tilde{P} = 0,\ \text{rank}\ \|\tilde{P}(x,y)\| = 2n - k';\quad 0 \le k' < 2n,\quad \forall (x,y) \in TM$$

In the adapted basis $X_A = \{\delta_i, \dot{\partial}_i\}$, $A = \overline{(1,2n)}$, $i = \overline{(1,n)}$, $\tilde{P}$ can be represented by:

$$(5.2)\qquad \tilde{P} = \overset{1}{P}{}^i_j\, \delta_i \otimes dx^j \otimes \overset{2}{P}{}^i_j\, \delta_i \otimes \delta y^j + \overset{3}{P}{}^i_j\, \dot{\partial}_i \otimes dx^j + \overset{4}{P}{}^i_j\, \dot{\partial}_i \otimes \delta y^j ,$$

where $\overset{\alpha}{P}{}^i_j$ $(\alpha = 1,2,3,4)$ are Finsler tensor fields on M.

Hence, we have

$$(5.3)\qquad \tilde{P}(\delta_j) = \overset{1}{P}{}^i_j\, \delta_i + \overset{3}{P}{}^i_j\, \dot{\partial}_i ,\qquad \tilde{P}(\dot{\partial}_j) = \overset{2}{P}{}^i_j\, \delta_i + \overset{4}{P}{}^i_j\, \dot{\partial}_i ,$$

and the condition (5.1) is equivalent with

$$(5.4)\quad \begin{cases} \left(\overset{1}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{1}{P}{}^s_j + \left(\overset{1}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{3}{P}{}^s_j - \overset{1}{P}{}^i_j = 0 , \\ \left(\overset{1}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{2}{P}{}^s_j + \left(\overset{1}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{4}{P}{}^s_j - \overset{2}{P}{}^i_j = 0 , \\ \left(\overset{3}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{1}{P}{}^s_j + \left(\overset{3}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{3}{P}{}^s_j - \overset{3}{P}{}^i_j = 0 , \\ \left(\overset{3}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{2}{P}{}^s_j + \left(\overset{3}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{4}{P}{}^s_j - \overset{4}{P}{}^i_j = 0 . \end{cases}$$

Also, we suppose that the components of $\tilde{P}$ fulfill conditions such as

$$H = \tilde{P}^2 , \qquad V = - \tilde{P}^2 + I , \tag{5.5}$$

in order to be orthogonal projectors and supplementary.

If $p^i_j(x,y)$ is a $p(3,-1)$-Finsler structure of index k on M, then on TM, in the presence of a non-linear connection, N, we have some special cases:

$$\begin{aligned} \overset{1}{\tilde{P}} &= p^i_j \ \delta_i \otimes dx^j + p^i_j \ \dot{\partial}_i \otimes \delta y^j , \\ \overset{2}{\tilde{P}} &= p^i_j \ \delta_i \otimes dx^j - p^i_j \ \dot{\partial}_i \otimes \delta y^j , \\ \overset{3}{\tilde{P}} &= p^i_j \ \delta_i \otimes dy^j + p^i_j \ \dot{\partial}_i \otimes d x^j . \end{aligned} \tag{5.6}$$

The tensor fields $\overset{\alpha}{\tilde{P}}$ $(\alpha = 1,2,3)$ given by (5.6) are $\tilde{P}(3,-1)$-structures of a special type on TM. Indeed, the conditions (2.2) and (2.3) being fulfiled for $p^i_j(x,y)$, we obtain

$$\begin{cases} \overset{\alpha}{\tilde{P}}{}^3 - \tilde{P} = o, \quad \operatorname{rank} \| \overset{\alpha}{\tilde{P}}(x,y) \|_{HTM} = \operatorname{rank} \| \overset{\alpha}{\tilde{P}}(x,y) \|_{VTM} = n - k \\ H = \overset{\alpha}{\tilde{P}}{}^2 = h^i_j \ \delta_i \otimes dx^j + h^i_j \ \dot{\partial}_i \otimes \delta y^j , \\ V = - \overset{\alpha}{\tilde{P}}{}^2 + I = v^i_j \ \delta_i \otimes dx^j + v^i_i \ \dot{\partial}_i \otimes \delta y^j , \quad \forall \ \alpha = 1,2,3 , \end{cases} \tag{5.7}$$

where $h^i_j = p^i_h \ p^h_j$, $v^i_j = - p^i_h \ p^h_j + \delta^i_j$.

Then (5.6) determines $\tilde{P}(3,-1)$-structures on TM by the lifting of one $p(3,-1)$-Finsler sturctures from M to the space total TM of a tangent budle $T(M)$.

We remark that the Nijenhuis tensor of $\tilde{P} \in \tau^1_1(TM)$ is given by

$$\tilde{N}(X,Y) = [\tilde{P}X, \tilde{P}Y] - \tilde{P}[\tilde{P}X, Y] - \tilde{P}[X, \tilde{P}Y] + H[X,Y] , \quad ; \tag{5.8}$$

and the integrability condition of a $\tilde{P}(3,-1)$-structures is $\tilde{N}(X,Y) = o$, $\forall \ X,Y \in (TM)$. It is sufficient to calculate $\tilde{N}(\delta_j, \delta_k)$, $N(\delta_j, \dot{\partial}_k)$ and $N(\dot{\partial}_j, \dot{\partial}_k)$ and we can determine $\tilde{N}(X,Y)$.

6. The integrability of the $p(3,-1)$-Finsler structures :

Let N be a non-linear connection of $T(M)$. Then the $p(3,-1)$-Finsler structure on the base manifold M is lifted to a $\tilde{P}(3,-1)$-structure on $T(M)$ in three manner (5.6). The values of the Finsler components of $\tilde{P}$ from (5.2) are given in the following table:

$\tilde{P}$	$\overset{1}{P}{}^i_j$	$\overset{2}{P}{}^i_j$	$\overset{3}{P}{}^i_j$	$\overset{4}{P}{}^i_j$
$\overset{1}{\tilde{P}}$	p^i_j	0	0	p^i_j
$\overset{2}{\tilde{P}}$	p^i_j	0	0	$-p^i_j$
$\overset{3}{\tilde{P}}$	0	p^i_j	p^i_j	0

We remark the following relations:

$$\begin{cases} \overset{1}{\tilde{P}}(\delta_j) = p^i_j\,\delta_i, & \overset{2}{\tilde{P}}(\delta_j) = p^i_j\,\delta_i, \quad \overset{3}{\tilde{P}}(\delta_j) = p^i_j\,\dot{\partial}_i\,, \\ \overset{1}{\tilde{P}}(\dot{\partial}_j) = p^i_j\,\dot{\partial}_i, & \overset{2}{\tilde{P}}(\dot{\partial}_j) = -p^i_j\,\dot{\partial}_i, \quad \overset{3}{\tilde{P}}(\dot{\partial}_j) = p^i_j\,\delta_i\,. \end{cases}$$

Definition 6.1. *A $p(3,-1)$ structure of index k on a differentiable manifold M is called an integrable of type I, II or III with respect to the non-linear connection N, if the corresponding lifted $\overset{1}{\tilde{P}}(3,-1)$, $\overset{2}{\tilde{P}}(3,-1)$, or $\overset{3}{\tilde{P}}(3,-1)$-structure are integrable.*

We characterize these cases of integrability, using only the invariants of the group G_p.

Theorem 6.1. *The $p(3,-1)$-Finsler structure $p^i_j(x,y)$, (2.1) is an integrable of type I, if and only if the following invariant Finsler tensor fields vanish:*

$$\overset{1}{T}{}^{i}_{jk} = o\ ,\quad \overset{1}{R}{}^{i}_{jk} = o\ ,\quad \overset{1}{C}{}^{i}_{jk} = o\ ,\quad \overset{1}{P}{}^{i}_{jk} = o\quad (\Rightarrow \overset{1}{S}{}^{i}_{jk} = o)\ . \tag{6.2}$$

Proof. The $p(3,-1)$-Finsler structure is an integrable of type I if and only if $\tilde{N}(X,Y) = 0$ for $\overset{1}{\tilde{P}}$. But $N(X,Y) = 0\ \forall\ X,Y \in \mathfrak{X}(TM)$ is equivalent to

$$\tilde{N}(\delta_j,\delta_k) = o\ ,\quad \tilde{N}(\delta_j,\dot{\partial}_k) = o\ ,\quad \tilde{N}(\dot{\partial}_j,\dot{\partial}_k) = o$$

which are equivalent to (6.2). In this case, because $\overset{1}{C}{}^{i}_{jk} = o$ we have $\overset{1}{S}{}^{i}_{jk} = o$.

In the same way we can prove

Theorem 6.2. *The $p(3,-1)$-Finsler structure $p^i_j(x,y)$, (2.1), is an integrable of type II, if and only if the following ivariant Finsler tensor fields vanish:*

$$\overset{1}{T}{}^{i}_{jk} = 0\ ,\quad \overset{2}{R}{}^{i}_{jk} = 0\ ,\quad \overset{2}{C}{}^{i}_{jk} = 0\ ,\quad \overset{2}{P}{}^{i}_{jk} = 0\ ,\quad \overset{1}{S}{}^{i}_{jk} = 0\ . \tag{6.3}$$

Theorem 6.3. *The $p(3,-1)$-Finsler structure $p^i_j(x,y)$, (2.1) is an integrable of type III, if and only if the following invariant Finsler tensor fields vanish:*

$$\overset{3}{T}{}^{i}_{jk} = 0\ ,\quad \overset{3}{R}{}^{i}_{jk} = 0\ ,\quad \overset{3}{C}{}^{i}_{jk} = 0\ ,\quad \overset{3}{P}{}^{i}_{jk} = 0\ ,\quad \tilde{T}^{i}_{jk} = 0\ ,\quad \overset{3}{S}{}^{i}_{jk} = 0\ . \tag{6.4}$$

Concluding by Theorems 6.1, 6.2 and 6.3 we obtain:

Theorem 6.4. *The $p(3,-1)$-Finsler structure $p^i_j(x,y)$, (2.1), is an integrable of type I, II or III if and only if the invariants of the group G_p have the values given in the following table:*

Type of integrability	Characterization by invariants
I	$\overset{1}{T}{}^{i}_{jk} = 0,\ \overset{1}{R}{}^{i}_{jk} = 0,\ \overset{1}{C}{}^{i}_{jk} = 0,\ \overset{1}{P}{}^{i}_{jk} = 0,\ \overset{1}{S}{}^{i}_{jk} = 0\ .$
II	$\overset{1}{T}{}^{i}_{jk} = 0,\ \overset{2}{R}{}^{i}_{jk} = 0,\ \overset{2}{C}{}^{i}_{jk} = 0,\ \overset{2}{P}{}^{i}_{jk} = 0,\ \overset{1}{S}{}^{i}_{jk} = 0\ .$
III	$\overset{3}{T}{}^{i}_{jk} = 0,\ \overset{3}{R}{}^{i}_{jk} = 0,\ \overset{3}{C}{}^{i}_{jk} = 0,\ \overset{3}{P}{}^{i}_{jk} = 0,\ \overset{3}{S}{}^{i}_{jk} = 0,\ \tilde{T}^{i}_{jk} = 0$

7. (h,v)-Walker structure on the total space of a vector bundle

Let $\xi = (E,\pi,M)$ a vector bundkle of the class C^{∞}, let N be a non-linear connection on E and let us denote by HE and VE the complementary horizontal and vertical distribution:

$$T_{(x,y)}E = H_{(x,y)}E + V_{(x,y)}E\,, \quad \forall\,(x,y) \in E\,.$$

Let $l(x,y)$ be an $l(3,-1)$-structure of index k_1 on the distribution HE and let $m(x,y)$ be an $m(3,-1)$-structure of index k_2 on the distribution VE. For any $(x,y) \in E$ we have

$$HE = H_hE + H_vE \qquad VE = V_hE + V_vE\,. \tag{7.1}$$

We denote by $\underset{1}{h}$, $\underset{1}{v}$ and respectively $\underset{2}{h}$, $\underset{2}{v}$, the supplementary projectors on the distribution (7.1).

Let $\{\delta_i, \dot{\partial}_a\}$, $\{dx^i, \delta y^a\}$ be the adapted basis of N and VE.

Under these conditions we can consider on E the aggregate tensor field of type $(1,1)$ given by

$$P = l^i_j(x,y)\delta_i \otimes dx^j + m^a_b(x,y)\,\dot{\partial}_a \otimes \delta y^b\,. \tag{7.2}$$

Because (2.2) and (2.3), § 2, are fulfilled for l, $\underset{1}{h}$, $\underset{1}{v}$ and respectively m, $\underset{2}{h}$, $\underset{2}{v}$ we have from (7.2) the following relations:

$$P^3 - P = 0\,, \quad \operatorname{rank}\|P\|_{HE} = k_1, \quad \operatorname{rank}\|P\|_{VE} = k_2\,. \tag{7.3}$$

We put $H = P^2$, $V = -P^2 + I$ and we obtain

$$\begin{cases} H = \underset{1}{h}{}^i_j(x,y)\,\delta_i \otimes dx^j + \underset{2}{h}{}^a_b(x,y)\,\dot{\partial}_a \otimes \delta y^b\,, \\[2ex] V = \underset{1}{v}{}^i_j(x,y)\,\delta_i \otimes dx^j + \underset{2}{v}{}^a_b(x,y)\,\dot{\partial}_a \otimes \delta y^b\,, \end{cases} \tag{7.4}$$

An elementary calculus shows us that

$$\begin{cases} HP = PH = P\,, & VP = PV = 0\,, \\[1ex] P^2H = H\,, & P^2V = 0\,. \end{cases} \tag{7.5}$$

This means that (7.2) is a $P(3,-1)$-structure on E.

Definition 7.1. *We shall call the structure given by (7.2) the (h,v)-Walker structure on the total space E of a vector bundle ξ.*

Definiton 7.2. *A d-connection D on E with coefficients* $D\Gamma = \left[L^i_{jk},\ L^a_{bk},\ C^i_{jc},\ C^a_{bc}\right]$ *is called a Walker d-connection or compatible with the (h,v)-Walker structure $P \in \tau^1_1(E)$, (6.2), if*

$$l^i_{j|k} = 0, \quad m^a_{b|k}, \quad l^i_j\Big|_c = 0, \quad m^a_b\Big|_c = 0, \tag{7.6}$$

where $|$ and $\Big|$ denotes h-and v-covariant derivatives with respect to $D\Gamma$.

Theorem 7.1. *Let* $D\mathring{\Gamma} = \left[\mathring{L}^i_{jk},\ \mathring{L}^a_{bk},\ \mathring{C}^i_{jc},\ \mathring{C}^a_{bc}\right]$ *be a fixed d-connection. Then, the d-connection $D\dot{\Gamma}$ given by*

$$\begin{cases} \dot{L}^i_{jk} = \mathring{L}^i_{jk} + \dfrac{1}{2}\left\{l^i_s\ l^s_j\overset{\circ}{|}_k - \underset{1}{v}{}^i_j\overset{\circ}{|}_k + 3\underset{1}{v}{}^i_s\ \underset{1}{v}{}^s_j\overset{\circ}{|}_k\right\}, \\ \dot{L}^a_{bk} = \mathring{L}^a_{bk} + \dfrac{1}{2}\left\{m^a_f\ m^f_b\overset{\circ}{|}_k - \underset{2}{v}{}^a_b\overset{\circ}{|}_k + 3\underset{2}{v}{}^a_f\ \underset{2}{v}{}^f_b\overset{\circ}{|}_k\right\}, \\ \dot{C}^i_{jc} = \mathring{L}^i_{jc} + \dfrac{1}{2}\left\{l^i_s\ l^s_j\overset{\circ}{\Big|}_c - \underset{1}{v}{}^i_j\overset{\circ}{\Big|}_c + 3\underset{1}{v}{}^i_s\ \underset{1}{v}{}^s_j\overset{\circ}{\Big|}_c\right\}, \\ \dot{C}^a_{bc} = \mathring{C}^a_{bc} + \dfrac{1}{2}\left\{m^a_f\ m^f_b\overset{\circ}{\Big|}_c - \underset{2}{v}{}^a_b\overset{\circ}{\Big|}_c + 3\underset{2}{v}{}^a_f\ \underset{2}{v}{}^f_b\overset{\circ}{\Big|}_c\right\}, \end{cases} \tag{7.7}$$

is a Walker d-connection, where $\overset{\circ}{|}$ and $\overset{\circ}{\Big|}$ are the h-and v-covariant derivatives with respect to $D\mathring{\Gamma}$.

If we take in (6.7) for $D\mathring{\Gamma}$ a Berward connection

$$\mathring{L}^a_{bk} = \dot{\partial}_b N^a_k, \quad \mathring{C}^i_{jc} = 0, \tag{7.8}$$

the we have

Theorem 7.2. *Let $D\mathring{\Gamma}$ be a fixed Berward connection. Then, the d-connection $D\overset{\gamma}{\Gamma}$ given by*

$$\begin{cases} \overset{\gamma}{L}{}^i_{jk} = \mathring{L}^i_{jk} + \dfrac{1}{2}\left\{l^i_s\ l^s_j\overset{\circ}{|}_k - \underset{1}{v}{}^i_j\overset{\circ}{|}_k + 3\underset{1}{v}{}^i_s\ v^s_j\overset{\circ}{|}_k\right\} \quad (= \dot{L}^i_{jk}), \\ \overset{\gamma}{L}{}^a_{bk} = \dot{\partial}_b N^a_k + \dfrac{1}{2}\left\{m^a_f\left[\delta_k m^f_b + m^d_b\ \dot{\partial}_d\ N^f_k - m^f_d\ \dot{\partial}_b\ N^d_k\right] - \right. \\ \qquad \left. - \left[\delta^a_f - 3\ \underset{2}{v}{}^a_f\right]\left[\delta_k\ \underset{2}{v}{}^f_b + \underset{2}{v}{}^d_b\ \dot{\partial}_d\ N^f_k - \underset{2}{v}{}^f_d\ \dot{\partial}_b\ N^d_k\right]\right\}, \\ \overset{\gamma}{C}{}^i_{jc} = \dfrac{1}{2}\left\{l^i_s\ \dot{\partial}_c\ l^s_j - \left[\delta^i_s - 3\underset{1}{v}{}^i_s\right]\dot{\partial}_c\ \underset{1}{v}{}^s_j\right\}, \\ \overset{\gamma}{C}{}^a_{bc} = \mathring{C}^a_{bc} + \dfrac{1}{2}\left\{m^a_f\ m^f_b\overset{\circ}{\Big|}_c - \underset{2}{v}{}^a_b\overset{\circ}{\Big|}_c + 3\underset{2}{v}{}^a_f\ \underset{2}{v}{}^f_b\overset{\circ}{\Big|}_c\right\} \quad (= \dot{C}^a_{bc}), \end{cases} \tag{7.9}$$

is a Walker d-connection, with respect to $p \in \tau^1_1(E)$, (7.2).

Remark 7.1. In (7.9) the coefficients $\overset{\gamma}{L}{}^a_{bk}$, and $\overset{\gamma}{C}{}^i_{jc}$ of the Walker d-connection $D\overset{\gamma}{\Gamma}$ depend on $N^i_j(x,y)$ $l^i_j(x,y)$ and $m^a_b(x,y)$, only, that is, they are canonically introduced.

If we denote Obata's operators (2.5), 2, of $l^i_j(x,y)$ and $m^a_b(x,y)$ by $\underset{1}{A}{}^{rs}_{ij}$, $\underset{2}{A}{}^{rs}_{ij}$ and respectively $\underset{1}{B}{}^{cd}_{ab}$, $\underset{2}{B}{}^{cd}_{ab}$, it is easy to prove:

Theorem 7.3. *The set of all the Walker d-connections with respect to the Walker structure $P \in \tau^1_1(E)$, (7.2),* is given by

$$(7.10)\quad \begin{cases} L^i_{jk} = \overset{\bullet}{L}{}^i_{jk} + \underset{1}{A}{}^{is}_{rj} X^r_{sk}, & L^a_{bk} = \overset{\bullet}{L}{}^a_{bk} + B^{af}_{db} X^d_{fk}, \quad \forall X^i_{jk} \in \tau^{10}_{20}, \quad X^a_{bk} \in \tau^{01}_{11}, \\ C^i_{jc} = \overset{\bullet}{C}{}^i_{jc} + \underset{1}{A}{}^{is}_{rj} Y^r_{sk}, & C^a_{bc} = \overset{\bullet}{C}{}^a_{bc} + B^{af}_{db} Y^d_{fc}, \quad \forall Y^i_{jc} \in \tau^{10}_{11}, \quad Y^a_{bc} \in \tau^{01}_{02}, \end{cases}$$

The transformations of Walker d-connections $D\Gamma(N) \to D\Gamma(\bar{N})$ with the same non-linear connection N are given by

$$(7.11)\quad \begin{cases} \bar{L}^i_{jk} = L^i_{jk} + \underset{1}{A}{}^{is}_{rj} X^r_{sk}, & \bar{L}^a_{bk} = L^a_{bk} + \underset{1}{B}{}^{af}_{db} X^d_{fk}, \\ \bar{C}^i_{jc} = C^i_{jc} + \underset{1}{A}{}^{is}_{rj} Y^r_{sc}, & \bar{L}^a_{bc} = L^a_{bc} + \underset{1}{B}{}^{af}_{db} X^d_{fc}, \end{cases}$$

Hence,

Theorem 7.4. *The set of all transformations (7.11) and the mapping product is an Abelian group G_W isommorphic with the additive group of a d-tensor field of the form*

$$\left(\underset{1}{A}{}^{is}_{rj} X^r_{sk},\ \underset{1}{B}{}^{af}_{db} X^d_{fk},\ \underset{1}{A}{}^{is}_{rj} Y^r_{sc},\ \underset{1}{B}{}^{af}_{db} Y^d_{fc} \right).$$

Theorem 7.5. *The following d-tensor fields are invariants by the action of the group G_W :*

$$(7.12)\quad \begin{cases} T(P)^i_{jk} = \underset{1}{h}{}^i_m T^m_{jk} + l^r_j l^s_k T^i_{rs} - \left(l^r_j T^m_{rk} + l^s_k T^m_{js} \right) l^i_m \quad (= \overset{1}{T}(l)^i_{jk}), \\ R(P)^a_{jk} = \underset{2}{h}{}^a_d R^d_{jk} + l^r_j l^s_k R^a_{rs} - \left(l^r_j R^d_{rk} + l^s_k R^d_{js} \right) m^a_c\,. \\ C(P)^i_{jk} = \underset{1}{h}{}^i_m C^m_{jc} + l^r_j m^g_c C^i_{rg} - \left(l^r_j C^m_{rc} + m^g_c C^m_{jg} \right) l^i_m\,, \\ P(P)^a_{jc} = \underset{2}{h}{}^a_d P^d_{jc} + l^r_j m^g_c P^a_{rg} - \left(l^r_j P^d_{rc} + m^g_c P^d_{jg} \right) m^a_d\,, \\ S(P)^a_{bc} = \underset{2}{h}{}^a_d S^d_{bc} + m^f_b m^g_c S^a_{fg} - \left(m^f_b S^d_{fc} + m^g_c S^d_{bd} \right) m^a_d \quad (= \overset{1}{S}(m)^a_{bc}). \end{cases}$$

Theorem 7.6. *The d-tensor fields* $T(P)^i_{jk}$, $S(P)^a_{bc}$ *vanish if and only if there exists on E an h-and v-semi-symmetric Walker d-connection* $D\Gamma(N)$.

The integrability condition of the (h,v)-Walker stucture $P \in \tau^1_1(E)$, (6.2), is $N(P)(X,Y) = o$, $\forall\ X,Y \in \mathcal{X}(E)$. Taking into account that we have the following relationships:

$$(7.13) \qquad P(\delta_j) = l^i_j\ \delta_i\ , \quad P(\dot\partial_b) = m^a_b\ \dot\partial_a\ ,$$

we obtain

Theorem 7.7. *The (h,v)-Walker structure* $P \in \tau^1_1(E)$, (6.2), *is integrable if and only if the invariants* (7.12) vanish:

$$(7.14) \qquad T(P)^i_{jk} = o, \quad R(P)^a_{jk} = o, \quad C(P)^i_{jc} = o, \quad P(P)^a_{jc} = o, \quad S(P)^a_{bc} = o\ .$$

Remark 7.2. If $E = TM$, the considerations from this paragraph remain valid.

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Rezime

$p(3,-1)$ - FINSLEROVE STRUKTURE I NJIHOVI LIFTOVI

Na diferencijalnoj mnogostrukosti M kao i na TM je definisana $p(3,-1)$ odnosno $P(3,-1)$ Finslerova struktura. Odredeni su koeficijenti koneksije koje su saglasne sa tim strukturama, dati su uslovi integrabilnosti. Nadene su invarijante grupe transformacije Finslerove koneksije i dati su neki specijalni slučajevi.

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SOME THEOREMS ON CONFORMALLY QUASI-RECURRENT MANIFOLDS

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Abstract

A conformally quasi-recurrent manifold is an n-dimensional ($n > 3$) Riemannian manifold whose conformal curvature tensor satisfies the condition (1.2), where ∇ is the operator of covariant differentiation. It is proved that if a_i is a gradient vector field, such a manifold can be conformally related to the conformally symmetric one (i.e. to the manifold satisfying $\nabla_s C_{hijk} = 0$). Using this fact, it is proved that many properties of conformally symmetric manifolds can be generalized in such a manner that they hold good for conformally quasi-recurrent manifolds too (in which a_i is a gradient vector field). Also, some properties of general quasi-recurrent manifolds are obtained.

1. Introduction

A conformally quasi-reccurent manifold has been defined in [3] as an n-dimensional ($n > 3$) Riemannian manifold M with a (possibly indefinite) metric g whose conformal curvature tensor

$$\begin{aligned} C^h_{ijk} &= R^h_{ijk} - \frac{1}{n-2}\left(g_{ij}R^h_k - g_{ik}R^h_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}\right) \\ &\quad + \frac{R}{(n-1)(n-2)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) \end{aligned} \tag{1.1}$$

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satisfies the condition

$$(1.2)\qquad \nabla_s C_{hijk} = 2a_s C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs}$$

where

$$C_{hijk} = g_{ht} C^t_{ijk} ,$$

a_s is a vector field and R^h_{ijk}, R_{ij}, R and ∇ denote the curvature tensor, Ricci tensor, scalar curvature and covariant differentiation respectively.

If

$$(1.3)\qquad \nabla_s C_{hijk} = 0,$$

M is said to be conformally symmetric. If, however,

$$\nabla_s C_{hijk} = a_s C_{hijk} ,$$

M is said to be conformally recurrent. Therefore, a conformally symmetric manifold is a special case of both the conformally recurrent and the conformally quasi-recurrent one.

M is said to be essentially conformally symmetric if it satisfies (1.3) but is neither conformally flat nor locally symmetric. These manifolds have been investigated in detail in [1], [2], [4] and [5]. Among other things it is proved there that any essentially conformally symmetric manifold satisfies the following relations:

(a) $\nabla_j R_{ik} = \nabla_k R_{ij}$;

(b) $R_{aj} R^a_{ikl} + R_{ak} R^a_{ilj} + R_{al} R^a_{ijk} = 0$;

(c) $R_{aj} C^a_{ikl} + R_{ak} C^a_{ilj} + R_{al} C^a_{ijk} = 0$;

(d) $\nabla_s R_{aj} C^a_{ikl} + \nabla_s R_{ak} C^a_{ilj} + \nabla_s R_{al} C^a_{ijk} = 0$;

(e) $R = 0$;

(f) $R_{ia} C^a_{jkl} = 0$;

(g) $C_{lmh}{}^a C_{aijk} + C_{lmi}{}^a C_{hajk} + C_{lmj}{}^a C_{hiak} + C_{lmk}{}^a C_{hija} = 0$;

(h) $\{R_{hl} C_{mijk} + R_{il} C_{hmjk} + R_{jl} C_{himk} + R_{kl} C_{hijm}\} - \{m/l\} = 0$, where in the expression $\{\ldots\} - \{m/l\}$, $\{m/l\}$ means the first bracket in which m and l interchange their places;

(i) $R^a_i R_{aj} = 0$;

(j) $R_{ij} R_{hk} - R_{hj} R_{ik} = F C_{hijk}$, for some function F.

The purpose of this paper is to find relations correspondig to (a)-(j) for a conformally quasi-recurrent manifold. In §2 we shall obtain the relations corresponding to (a) and (d) and prove that (b) and (c) hold good. In §3 and §4 we deal with conformally quasi-recurrent manifolds in which the vector field a_i is a gradient. In §3 we shall prove that such a manifold can be conformally related to the conformally symmetric one. Using this, we shall find in §4 the relations corresponding to (e)-(j).

2. General case

It is well known that the conformal curvature tensor satisfies the relations

(2.1) $$C^h_{ijk} = - C^h_{ikj} ,$$

(2.2) $$C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0 ,$$

(2.3) $$C^a_{ajk} = C^a_{jak} = C^a_{jka} = 0 ,$$

(2.4) $$C_{hijk} = - C_{ihjk} , \quad C_{hijk} = C_{jkhi} .$$

Now, we shall prove

Theorem 1. *Any conformally quasi-recurrent manifold satisfies*

(2.5) $$\nabla_k \Pi_{ij} = \nabla_j \Pi_{ik} = 0 ,$$

where

(2.6) $$\Pi_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij} .$$

Relation (2.5) corresponds to relation (a) of §1.

Proof. Transvecting (1.2) with g^{hk} and taking into account (2.3), we get

$$a^h C_{sijh} + a^h C_{hijs} = 0$$

which, because of (2.1) and (2.4), can be rewritten in the form :

$$a^h C_{hijs} = a^h C_{hjsi} . \tag{2.7}$$

On the other hand, in view of (2.2), we have

$$a^h (C_{hijs} + C_{hjsi} + C_{hsij}) = 0 ,$$

which, using (2.7), reduces to

$$a^h C_{hijs} = 0 . \tag{2.8}$$

Therefore, transvecting (1.2) with g^{sh}, we get

$$\nabla_s C^s_{ijk} = 0 . \tag{2.9}$$

On the other hand, differentiating (1.1) covariantly, we have

$$\nabla_s C^h_{ijk} = \nabla_s R^h_{ijk} - \frac{1}{(n-2)} (g_{ij} \nabla_s R^h_k - g_{ik} \nabla_s R^h_j + \delta^h_k \nabla_s R_{ij} - \delta^h_j \nabla_s R_{ik}) +$$

$$+ \frac{\nabla_s R}{(n-1)(n-2)} (\delta^h_k g_{ij} - \delta^h_j g_{ik}) ,$$

from which, contracting with respect to s and h and taking into account that

$$\nabla_s R^s_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} \quad \text{and} \quad \nabla_s R^s_k = \frac{1}{2} \nabla_k R ,$$

we get

$$\nabla_s C^s_{ijk} = \frac{n-3}{n-2} \left[\nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (g_{ij} \nabla_k R - g_{ik} \nabla_j R) \right] .$$

This, together with (2.9) and (2.6) leads to (2.5), because of $n > 3$.

Theorem 2. *For any conformally quasi-recurrent manifold, relations (b) and (c) of §1 hold good.*

Proof. Differentiating (2.5) covariantly, we get

$$\nabla_l \nabla_k \Pi_{ij} - \nabla_l \nabla_j \Pi_{ik} = 0 . \tag{2.10}$$

Permuting in (2.10) the indices *j*, *k* and *l* cyclically, adding the resulting equations to (2.10) and using the Ricci identity, we obtain

$$(2.11)\qquad \Pi_{aj}R^a_{ikl} + \Pi_{ak}R^a_{ilj} + \Pi_{al}R^a_{ijk} = 0\ .$$

Substituting (2.6) into this relation, we get (b) of §1.

On the other hand, (1.1) can be written in the form

$$R^h_{ijk} = C^h_{ijk} + \frac{1}{n-2}\left[g_{ij}\left(R^h_k - \frac{R}{2(n-1}\delta^h_k\right) - g_{ik}\left(R^h_j - \frac{R}{2(n-1}\delta^h_j\right) + \right.$$

$$\left. + \delta^h_k\left(R_{ij} - \frac{R}{2(n-1)}g_{ij}\right) - \delta^h_j\left(R_{ik} - \frac{R}{2(n-1}g_{ik}\right)\right]$$

or, using (2.6), in the form

$$R^h_{ijk} = C^h_{ijk} + \frac{1}{n-2}\left(g_{ij}\Pi^h_k - g_{ik}\Pi^h_j + \delta^h_k\Pi_{ij} - \delta^h_j\Pi_{ik}\right)$$

Substituting this into (2.11) and taking into account that Π_{ij} is a symmetric tensor, we get

$$\Pi_{aj}C^a_{ikl} + \Pi_{ak}C^a_{ilj} + \Pi_{al}C^a_{ijk} = 0\ .$$

Substituting (2.6) into this relation, we obtain (c) of §1.

Theorem 3. *Any conformally quasi-recurrent manifold satisfies*

$$(2.12)\qquad T_{saj}C^a_{ikl} + T_{sak}C^a_{ilj} + T_{sal}C^a_{ijk} = 0\ ,$$

where

$$T_{saj} = \nabla_s R_{aj} - R_{sa}a_j + g_{sa}a^tR_{tj}\ .$$

Relation (2.12) corresponds to (f) of §1.

Proof. Differentiating relation (c) of §1 covariantly, substituting (1.2) and using (c), we find

$$(\nabla_s R_{aj})C^a_{ikl} + (\nabla_s R_{ak})C^a_{ilj} + (\nabla_s R_{al})C^a_{ijk} +$$

$$+ a^t(R_{tj}C_{sikl} + R_{tk}C_{silj} + R_{tl}C_{sijk}) +$$

$$+ a_k(R_{aj}C^a_{isl} + R_{al}C^a_{ijs}) + a_l(R_{aj}C^a_{iks} + R_{ak}C^a_{isj}) +$$

$$+ a_j(R_{ak}C^a_{ils} + R_{al}C^a_{isk}) = 0 .$$

Using (c) once more, we obtain (2.12).

In the sequel, we shall need

Lemma *Any conformally quasi-recurrent manifold satisfies*

$$(2.13) \qquad (\nabla_r a_h)C^h_{ijk} = - a_h a^h C_{rijk} .$$

Proof. Differentiating (2.8) covariantly, we have

$$(\nabla_r a_h)C^h_{ijk} + a_h \nabla_r C^h_{ijk} = 0 .$$

Now, substituting (1.2) and using (2.8), we get (2.13).

3. Conformal change of a conformally quasi-recurrent manifold in which a_i is a gradient vector field

Now, let us suppose that for each point $x \in M$ of conformally quasi-recurrent manifold (M,g) there exists a neighbourhood U of x and a function f on U such that $a_i = \frac{\partial f}{\partial x^i}$. Let metrics $\bar{g}$ and g be conformally related such that

$$(3.1) \qquad \bar{g}_{ij} = e^{2f} g_{ij} , \qquad \bar{g}^{ij} = e^{-2f} g^{ij} .$$

Then, the Christoffel symbols of metrics $\bar{g}$ and g are related as follows:

$$\overline{\left\{ {k \atop ij} \right\}} = \left\{ {k \atop ij} \right\} + \delta^k_i a_j + \delta^k_j a_i - g_{ij} a^k ,$$

while the conformal curvature tensor is invariant:

$$(3.2) \qquad \bar{C}^h_{ijk} = C^h_{ijk} .$$

As for tensor $\bar{C}_{hijk}$, we have

$$(3.3) \qquad \bar{C}_{hijk} = e^{2f} C_{hijk} .$$

Let $\bar{\nabla}$ be the operator of covariant differentiation with respect to $\overline{\left\{ {k \atop ij} \right\}}$. Then, applying it to (3.2), and using (3.1), we get

$$\bar\nabla_s \bar C^h_{ijk} = \nabla_s C^h_{ijk}$$

$$(3.4) \qquad - 2 a_s C^h_{ijk} - a^h C_{sijk} - a_i C^h_{sjk} - a_j C^h_{isk} - a_k C^h_{ijs}$$

$$+ \delta^h_s a_r C^r_{ijk} + g_{is} a^r C^h_{rjk} + g_{js} a^r C^h_{irk} + g_{ks} a^r C^h_{ijr} ,$$

from which follows

$$(3.5) \qquad \bar\nabla_s \bar C^h_{ijk} = 0 ,$$

because of (1.2), (2.1), (2.4) and (2.8).

Thus, if (M, g) is conformally quasi-recurrent, then from $a_i = \dfrac{\partial f}{\partial x^i}$ and (3.1) it follows that $(M, \bar g)$ is conformally symmetric.

Consider, now, instead of (3.1), a new conformal change

$$\tilde g_{ij} = e^{2\varphi} g_{ij} , \quad \varphi = \varphi(x^1, \dots, x^n), \quad \frac{\partial \varphi}{\partial x^i} = b_i .$$

As in (3.4), we have

$$\tilde\nabla_s \tilde C^h_{ijk} = \nabla_s C^h_{ijk}$$

$$- 2 b_s C^h_{ijk} - b^h C_{sijk} - b_i C^h_{sjk} - b_j C^h_{isk} - b_k C^h_{ijs}$$

$$+ \delta^h_s b_r C^r_{ijk} + g_{is} b^r C^h_{rjk} + g_{js} b^r C^h_{irk} + g_{ks} b^r C^h_{ijr} .$$

Suppose that $(M, \tilde g)$ is conformally symmetric. Then, substituting $\tilde\nabla_s \tilde C^h_{ijk} = 0$ and (1.2) into the preceding relation, we get

$$2(a_s - b_s) C^h_{ijk} + (a^h - b^h) C_{sijk} + (a_i - b_i) C^h_{sjk} + (a_j - b_j) C^h_{isk}$$

$$(3.6) \qquad + (a_k - b_k) C^h_{ijs}$$

$$+ \delta^h_s b_r C^r_{ijk} + g_{is} b^r C^h_{rjk} + g_{js} b^r C^r_{irk} + g_{ks} b^r C^h_{ijr} = 0 .$$

Contracting (3.6) with respect to h and s and using (2.2), (2.3), (2.4) and (2.8), we obtain

$$(n - 3) b_s C^s_{ijk} = 0 \quad \text{i.e.} \quad b_s C^s_{ijk} = 0$$

because of $n > 3$. Thus, (3.6) can be written in the form

$$2\nu_s C_{hijk} + \nu_h C_{sijk} + \nu_i C_{hsjk} + \nu_j C_{hisk} + \nu_k C_{hijs} = 0\,, \tag{3.7}$$

where we have put

$$\nu_i = a_i - b_i\,.$$

But, it follows from (3.7) that either $\nu_i = 0$ or $C_{hijk} = 0$ (see [6], Lemma 3). Under our assumption, $C_{hijk} \neq 0$. Therefore, $\nu_i = 0$ i.e. $a_i = b_i$. Thus we have proved

Theorem 4. *A conformally quasi-recurrent manifold in which* a_i *is a gradient vector field, can always be locally conformally related to a conformally symmetric manifold.*

Conversely, if a conformally quasi-recurrent manifold can be conformally related to a conformally symmetric one, and if the corresponding conformal change is of the form (3.1), *then*

a) the vector field a_i *is locally a gradient and* $a_i = \dfrac{\partial f}{\partial x^i}$;

b) function f *satisfies the condition* $a_r \vec{C}^{r}_{ijk} = 0$.

Statement b) is an immediate consequence of (3.2) and (2.8).

4. Some properties of conformally quasi-recurrent manifolds in which a_i is a gradient vector field

Now, using Theorem 4 and the properties (e)-(j) of a conformally symmetric manifold, it is easy to find the corresponding properties of conformally quasi-recurrent manifolds in which a_i is a gradient vector field.

In all the theorems in this section, a conformally quasi-recurrent manifold means that one in which a_i is a gradient vector field.

As an immediate consequence of Theorem 4, (3.2), (3.3) and (g) of §1, we have

Theorem 5. *Let M be a conformally quasi-recurrent manifold. Then, relation* (g) holds good.

To obtain the other properties, we designate by $\vec{R}^{h}_{ijk}$ the cuvature tensor of a Riemannian space with metric (3.1), i.e. of that conformally symmetric manifold which is conformally related to the considered conformally quasi-recurrent one. Then,

$$(4.1)\qquad \bar{R}^{h}_{ijk} = R^{h}_{ijk} \quad \delta^{h}_{k}\alpha_{ji} - \delta^{h}_{j}\alpha_{ki} + g_{ij}\alpha^{h}_{k} - g_{ik}\alpha^{h}_{j} ,$$

where

$$(4.2)\qquad \alpha_{ji} = \nabla_{j}a_{i} - a_{i}a_{j} + \frac{1}{2}\, g_{ij}a_{t}a^{t} .$$

Contracting (4.1) with respect to k and h, we find

$$(4.3)\qquad \bar{R}_{ij} = R_{ij} + (n-2)\alpha_{ji} + g_{ij}\alpha^{t}_{t} .$$

Transvecting (4.3) with $\bar{g}^{ij}$, we get

$$\bar{R} = e^{-2f}\,[R + 2(n-1)\alpha^{t}_{t}]$$

from which, as a consequence of Theorem 4 and relation (e) of §1, we have

Theorem 6. *The scalar curvature of a conformally quasi-recurrent manifold has the form*

$$(4.4)\qquad R = -\,2(n-1)\,\alpha^{t}_{t} .$$

Using (4.4), we can rewrite (4.3) in the form

$$(4.5)\qquad \bar{R}_{ij} = \Pi_{ij} + (n-2)\alpha_{ij} .$$

Also, we can prove

Theorem 7. *Let M be a conformally quasi-recurrent manifold. Then M satisfies the relations.*

$$(4.6)\qquad \Pi_{it}C^{t}_{jkl} = \frac{n-2}{2}\, a_{t}a^{t}C_{ijkl} ;$$

$$(4.7)\qquad R_{it}C^{t}_{jkl} = -\nabla_{t}a^{t}C_{ijkl} ;$$

$$(4.8)\qquad R^{ab}C_{ajkb} = 0 ;$$

$$(4.9)\qquad \left\{\left[\Pi_{hl} + (n-2)\alpha_{hl}\right] C_{mijk} + \left[\,\Pi_{il} + (n-2)\alpha_{il}\right] C_{hmjk}\right.$$

$$+\left[\Pi_{ji} + (n-2)\alpha_{ji}\right] C_{himk} + \left[\Pi_{ki} + (n-2)\alpha_{ki}\right] C_{hijm} \Big\} - \Big\{ m/l \Big\} = 0;$$

(4.10)
$$\left[\Pi_{ia} + (n-2)\alpha_{ia}\right]\left[\Pi^a_j + (n-2)\alpha^a_j\right] = 0\ ;$$

(4.11)
$$\left[\Pi_{ij} + (n-2)\alpha_{ij}\right]\left[\Pi_{hk} + (n-2)\alpha_{hk}\right] - \left[\Pi_{hj} + (n-2)\alpha_{hj}\right]\left[\Pi_{ik} + (n-2)\alpha_{ik}\right]$$
$$= FC_{hijk} \quad \textit{for some function } F.$$

The relations (4.6) and (4.7) correspond to relation (*f*) of §1, (4.9) - to relation (*h*), (4.10) - to (*i*) and (4.11) - to (*j*).

Proof. In view of (2.8) and (2.13), we have

$$\alpha_{rt} C^t_{ijk} = -\frac{1}{2} a_t a^t C_{rijk}\ ,$$

so that, by virtue of (3.2) and (4.3), we find

(4.12)
$$\bar{R}_{it}\bar{C}^t_{jkl} = R_{it}C^t_{jkl} + \left(\alpha^t_t - \frac{n-2}{2} a_t a^t\right) C_{ijkl}\ .$$

Using (4.4), we get

$$\bar{R}_{it}\bar{C}^t_{jkl} = \Pi_{it}C^t_{jkl} - \frac{n-2}{2} a_t a^t C_{ijkl}\ .$$

Now, (4.6) follows from Theorem 4 and the relation $\bar{R}_{it}\bar{C}^t_{jkl} = 0$ (i.e. relation (*f*) of §1).

We can prove (4.7) in a similar manner. In fact, transvecting (4.2) with g^{ij}, we find

$$\alpha^t_t = \nabla_t a^t + \frac{n-2}{2} a_t a^t\ .$$

Substituting this into (4.12), we get

$$\bar{R}_{it}\bar{C}^t_{jkl} = R_{it}C^t_{jkl} + \nabla_t a^t C_{ijkl}\ .$$

Relation (4.8) is an immediate consequence of (4.7) and (2.3). The relations (4.9), (4.10) and (4.11) are consequences of Theorem 4, (4.5) and (*h*), (*i*) and (*j*) of §1.

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Rezime

NEKE TEOREME O KONFORMNO KVAZI-REKURENTNIM MNOGOSTRUKOSTIMA

Konformno kvazi-rekurentna mnogostukost je n-dimenziona ($n > 3$) Rimanova mnogostrukost čiji tenzor konformne krivine C_{hijk} zadovoljava uslov (1.2). Dokazano je da se takva mnogostrukost, ukoliko je a_i gradijentno vektorsko polje, može konformno preslikati na konformno simetričnu mnogostrukost (tj. na mnogostrukost koja zadovoljava uslov (1.3)). Koristeći tu činjenicu, dokazano je da se mnoge osobine konformno simetričnih mnogostrukosti mogu uopštiti tako da važe za one mnogostrukosti koje su konformno kvazi-rekurentne (a kod kojih je a_i gradijentno vektorsko polje). Takođe su dokazane i neke osobine opštih konformno kvazi-rekurentnih mnogostrukosti.

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Rezime

NEKE TEOREME O KONFORMNO KVAZI-REKURENTNIM MNOGOSTRUKOSTIMA

Konformno kvazi-rekurentne mnogostrukosti su n-dimenzionalne (n > 2) Rimanove mnogostrukosti čiji tenzor konformne krivine C_{hijk} zadovoljava uslov (1.2). Dokazano je da su takve mnogostrukosti, ukoliko je a_i [illegible] polje, [illegible] konformno [illegible] mnogostrukosti (1.1) [illegible] (2.2). [illegible] dokazano je da [illegible] mnogostrukosti [illegible] konformno kvazi-rekurentne [illegible] a_i [illegible] Takodje su dokazane i neke osobine [illegible] konformno kvazi-rekurentnih mnogostrukosti.

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A NOTE ABOUT A COVARIANT DERIVATIVE OF A HARMONIC VECTOR FIELD IN A RIEMANNIAN MANIFOLD

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Abstract

Betti numbers are related to the topology of the manifold. The one-dimensional Betti number is equal to the number of linearly independent harmonic vector fields on the manifold. Every harmonic vector field is a gradient vector field. Using that fact, we are getting some results about covariant derivative of a harmonic vector field.

Introduction

Let us consider a transformation of the metric tensor on an n-dimensional Riemannian space M

$$(0.1) \qquad \bar{g}_{ik} = e^{2\sigma} g_{ik}.$$

Then, the coefficients of their Riemannian connections are related

$$(0.2) \qquad \overline{\left\{ {i \atop jk} \right\}} = \left\{ {i \atop jk} \right\} + \pi_j \delta^i_k + \pi_k \delta^i_j - \pi^i g_{jk},$$

where $\overline{\{\ \}}$ are coefficients of the connection attached to metric $\bar{g}$ and $\{\ \}$ - coefficients of connection attached to the metric g. π_j is a gradient vector field and $\pi_j = \dfrac{\partial\sigma}{\partial x^j}$.

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Since this transformation is an "angle preserving" transformation, we call it a conformal transformation.

Under a conformal transformation of the metric, the curvature tensor will be transformed into

$$R^i_{jkl} = R^i_{jkl} + \delta^i_k \psi_{lj} - \delta^i_l \psi_{kj} + g_{jl}\psi^i_k - g_{jk}\psi^i_l, \tag{0.3}$$

where

$$\psi_{ij} = \nabla_i \pi_j - \pi_i \pi_j + \frac{1}{2}\pi_s \pi^s g_{ji} \; ; \quad \psi^j_i = g^{jk}\psi_{ik} \; ; \quad \pi^a = g^{ab}\pi_b ,$$

but the tensor

$$\bar{C}^i_{jkl} = \bar{R}^i_{jkl} + \frac{1}{n-2}(\delta^i_k \bar{R}_{jl} - \delta^i_l \bar{R}_{jk} + g_{jl}\bar{R}^i_k - g_{jk}\bar{R}^i_l) - \tag{0.4}$$

$$- \frac{\bar{R}}{(n-1)(n-2)}(\delta^i_k g_{jl} - \delta^i_l g_{jk})$$

will be invariant.

Conversely, the tensor (0.4) will be an invariant of a transformation (0.2) if and only if π_i is a gradient vector field.

If the vector π_i in expression (0.2) satisfies the condition

$$\nabla_k \pi_i = \pi_k \pi_i + \rho\, g_{ik} \tag{0.5}$$

(ρ is an arbitrary function), the invariant tensor of such a transformation will be

$$\bar{z}^i_{jkl} = \bar{R}^i_{jkl} + \frac{R}{n(n-1)}(\delta^i_k g_{jl} - \delta^i_l g_{jk}) . \tag{0.6}$$

Condition (0.5) is called a concircularity condition. Tensor (0.4) is called the conformal curvature tensor. Tensor (0.6) is called the concircular curvature tensor, because transformation (0.2) under condition (0.5) sends a geodesic circle into a geodesic circle. The symbols ∇ and $\bar{\nabla}$ denote a covariant differentiation with respect to connections $\{\ \}$ and $\overline{\{\ \}}$.

A vector field is called a harmonic vector field if it satisfies

$$\nabla_k v_i - \nabla_i v_k = 0 \tag{0.7}$$

and

$$\nabla_k v^k = 0 \; ; \quad v^k = g^{ka} v_a . \tag{0.8}$$

It is well known that the number of linearly independent harmonic vector fields in the manifold is equal to its one-dimensional Betti number. If the one-dimensional Betti number of the manifold is non-zero, then its Ricci curvature form cannot be a positive definite. If the Ricci curvature form in the direction of harmonic vector field vanishes, such a vector field is a parallel vector field.

1. A concircular transformation of the Riemannian connection with a harmonic generator

Suppose that the Riemannian manifold M has non-zero one-dimensional Betti number and the vector field π_i is one of its harmonic vector fields. Then, it is a gradient vector field, and for the curvature tensor of the connection (0.2), we will get the relation

$$\overline{R}_{ijkl} = R_{ijkl} + g_{ik}\psi_{lj} - g_{il}\psi_{kj} + g_{jl}\psi_{ki} - g_{jk}\psi_{li} \tag{1.1}$$

which is, in fact, relation (0.3) after lowering the superscript i. Contracting (1.1) by g^{il}, we get

$$\overline{R}_{jk} = R_{jk} + (2-n)\psi_{kj} - g_{jk}\psi_{il}g^{il}. \tag{1.2}$$

Transvecting (1.2) by g^{jk} we, get

$$\overline{R} = r + 2(1-n)\psi_k{}^k, \tag{1.3}$$

From (1.1), (1.2) and (1.3) there is the invariance of the tensor (0.4).

Now, the vector π_i is harmonic and

$$\psi_k{}^k = g^{jk}\psi_{kj} = \nabla_s\pi^s - \pi_s\pi^s + \frac{1}{2}n\pi_s\pi^s = \frac{n-2}{2}\pi_s\pi^s, \tag{1.4}$$

Then

$$\overline{R} = R - (n-1)(n-2)\pi_s\pi^s \tag{1.5}$$

and

$$\overline{R}_{jk} = R_{jk} - (n-2)(\nabla_k\pi_j - \pi_k\pi_j + \pi_s\pi^s g_{jk}) \tag{1.6}$$

or

$$\overline{R}_{jk} = R_{jk} - (n-2)\psi_{kj} - \frac{n-2}{2}\pi_s\pi^s g_{jk}.$$

Suppose, now, that the vector π_i satisfies the concircularity condition (0.5). Then,

(1.7)
$$\psi_{kj} = (\rho + \frac{1}{2}\pi_s\pi^s)g_{jk} ,$$

where ρ is the arbitrary function from the concircularity condition (0.5). Then,

(1.8)
$$\bar{R}_{ijkl} = R_{ijkl} + (2\rho + \pi_s\pi^s)(g_{ik}g_{lj} - g_{il}g_{kj})$$

(1.9)
$$\bar{R}_{jk} = R_{jk} - (n-1)(2\rho + \pi_s\pi^s)g_{jk}$$

and

(1.10)
$$\bar{R} = R - n(n-1)(2\rho + \pi_s\pi^s) .$$

As we have supposed earlier, the vector π_i is a harmonic vector field, and according to (0.8) and (0.5)

$$\rho = -\frac{\pi_s\pi^s}{n}$$

and

$$2\rho + \pi_s\pi^s = \frac{n-2}{n}\pi_s\pi^s .$$

Then, (1.8) can be transformed into

(1.8')
$$\bar{R}_{ijkl} = R_{ijkl} + \frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{jk}) ,$$

(1.9) can be transformed into

(1.9')
$$\bar{R}_{jk} = R_{jk} - \frac{(n-1)(n-2)}{n}\pi_s\pi^s g_{jk} ,$$

and (1.10) can be transformed into

(1.10')
$$\bar{R} = R - (n-1)(n-2)\pi_s\pi^s .$$

As the generator of connection (0.2) is a gradient and satisfies the concircularity condition, then the concircular curvature tensor of connection (0.2) has to be equal to the concircular curvature tensor of the Levi-Civita connection.

Using the formulae (1.8'), (1.9') and (1.10'), we can express the covariant components of the concircular curvature tensor Z

$$\bar{Z}_{ijkl} = R_{ijkl} + \frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{kj}) - \frac{R}{n(n-1)}(g_{ik}g_{lj} - g_{il}g_{kj})$$

$$+ \frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{kj}) = Z_{ijkl} + 2\frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{kj})$$

As the concircular curvature tensor is invariant under concircular transformations, we get

$$\frac{n-2}{n} \pi_s \pi^s (g_{ik} g_{lj} - g_{il} g_{kj})$$

or, if we presume dim $M > 2$,

$$\pi_s \pi^s = 0 \ . \tag{1.11}$$

So, we have proved

Theorem 1. *If the one-dimensional Betti number of the differentiable manifold M of a positive definite metric is non-zero, every harmonic vector field can serve as a generator of a connection of type (0.2), but none of these transformations is concircular.*

Corollary 1. *On an n-dimensional differentiable manifold M with a positive definite metric and a positive one-dimensional Betti number* B_1, *there does not exist any harmonic vector field which satisfies condition (0.5).*

Corollary 2. *If a harmonic vector field on a differentiable manifold M satisfies condition (0.5), its metric form cannot be a positive definite. Such a space can be an Einstein space if and only if such a harmonic vector field is a null-parallel vector field.*

2. Properties of a semi-symmetric metric connection with a harmonic generator

For a semi-symmetric metric connection

$$\Lambda^i_{jk} = \left\{ {i \atop jk} \right\} + \pi_j \delta^i_k - \pi^i g_{jk}$$

there hold

Proposition 1. *a) The conformal curvature tensor of connection (2.1) is equal to the conformal curvature tensor of the Levi-Civita connection.*

b) If the vector π_i *satisfies condition (0.5), the projective curvature tensor of connection (2.1) is equal to the projective curvature tensor of the Levi-Civita connection and the concircular curvature tensor of connection (2.1) is equal to the concircular curvature tensor of the Levi-Civita connection.*

Since, according to Corollary 1, no harmonic vector field can satisfy condition (0.5), we have

Corollary 3. *On a differentiable manifold M, with a positive definite metric form, if the vector* π_i *in expression* (2.1) *is harmonic, then connection* (2.1) *has neither geodesic lines nor geodesic circles in common with the Levi-Civita connection.*

3. Covariant derivative of a harmonic vector field

We have made it clear in 1. that the covariant derivative of a harmonic vector field on a differentiable manifold cannot be of the form (0.5). Now, we want to find, if it is possible, a totally general form of an expression which could represent a covariant derivative of a harmonic vector field.

Now, we know that if the vector π_i is harmonic, the concircular curvature tensor of connection (0.3) is not equal to the concircular curvature tensor of the Levi-Civita connection. We shall consider the tensor

$$D_{ijkl} = \bar{Z}_{ijkl} - Z_{ijkl} . \tag{3.1}$$

Transvecting (3.1) by g^{il}, we get

$$g^{il} D_{ijkl} = D_{jk} = \bar{R}_{jk} - R_{jk} - \frac{\bar{R}-R}{n} g_{jk} \tag{3.2}$$

Using the formulae (1.5) and (1.6), we can easily get

$$D_{jk} = -(n-2)(\nabla_k \pi_j - \pi_k \pi_j + \pi_s \pi^s) g_{jk} + \frac{(n-1)(n-2)}{n} \pi_s \pi^s g_{jk}$$

And finally

$$\nabla_k \pi_j = \pi_k \pi_j - \frac{1}{n} \pi_s \pi^s g_{jk} - \frac{1}{n-2} D_{jk} \tag{3.3}$$

We can get even more properties. For example,

$$D_s^{\ s} = \bar{R}_s^{\ s} - R_s^{\ s} - (\bar{R}-R) = 0$$

Finally, we have the next

Proposition 2. *A vector field* π_i *is harmonic if and only if*

$$g^{jk} \nabla_k \nabla_j \pi_i = R^i_j \pi^j \tag{3.4}$$

Now, we shall differentiate (3.3) one more time:

$$\nabla_i \nabla_k \pi_j = \pi_k \nabla_i \pi_j + \pi_j \nabla_i \pi_k - \frac{1}{n} g_{jk}(\pi_s \nabla_i \pi^s + \pi^s \nabla_i \pi_s) - \frac{1}{n-2} \nabla_i D_{jk} =$$

$$\pi_k \pi_i \pi_j - \frac{1}{n} \pi_k \pi g_{ij} - \frac{1}{n-2} \pi_k D_{ij} + \pi_j \pi_k \pi_i - \frac{1}{n} \pi \pi_j g_{ik} - \frac{1}{n-2} \pi_j D_{ik} -$$

$$- \frac{1}{n} g_{jk}(\pi_s \pi_i \pi^s - \frac{1}{n} \pi_i \pi - \frac{1}{n-2} \pi_s D^s_i + \pi_s \pi_i \pi_s - \frac{1}{n} \pi \pi_i - \frac{1}{n-2} \pi^3 D_{is}) -$$

$$- \frac{1}{n-2} \nabla_i D_{jk} =$$

$$= 2\pi_k \pi_i \pi_j - \frac{1}{n} \pi_k \pi g_{ij} - \frac{1}{n} \pi \pi_j g_{ik} - \frac{2}{n} \pi \pi_i g_{jk} + \frac{2}{n^2} \pi \pi_i g_{ik}$$

$$+ \frac{2}{n(n-2)} \pi^s D_{is} g_{jk} - \frac{1}{n-2} \pi_k D_{ij} - \frac{1}{n-2} \pi_j D_{ik} - \frac{1}{n-2} \overset{0}{\nabla}_i D_{jk}$$

After transvecting by g^{ik}, we get

$$\frac{(n-1)(n-2)}{n^2} \pi \pi_i - \frac{1}{n} \pi_k D^k_i - \frac{1}{n-2} g^{jk} \nabla_i D_{jk} = R_{ij} \pi^i$$

Transvecting again by π^i, we get

$$(3.5) \qquad \frac{(n-1)(n-2)}{n^2} \pi^2 - \frac{1}{n} D_{ab} \pi^a \pi^b - \frac{1}{n-2} \pi^j g^{ik} \nabla_i D_{jk} = R_{ij} \pi^i \pi^j$$

We have just proved

Theorem 2. On a Riemannian manifold with an indefinite Ricci form and a positive definite metric, the covariant derivative of a harmonic vector field is of the form (3.3). D_{jk} is a non-zero symmetric tensor field which is not proportional to the metric tensor and wich satisfies

(1) $D^s_s = 0$

(2) (3.5), where π denotes $\pi_s \pi^s$

(3) $\frac{(n-1)(n-2)}{n^2} \pi^2 - \frac{1}{n} D_{ab} \pi^a \pi^b - \frac{1}{n-2} \pi^j g^{ik} \nabla_i D_{jk} < 0$

(4) $g^{ik}(\nabla_i D_{jk} - \nabla_k D_{ji}) = 0.$

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Rezime

NOTA O KOVARIJANTNIM IZVODIMA HARMONIJSKOG VEKTORSKOG POLJA U RIMANOVOJ MNOGOSTRUKOSTI

U radu je izveden najopštiji oblik kovarijantnog izvoda harmonijskog vektorskog polja. Dokazano je da harmonijski vektor ne može zadovoljiti uslov koncirkularnosti ako je metrika pozitivno definitna. Ako je metrika indefinitna, potprostor generisan harmonijskim vektorima je Ajnštajnov ako i samo ako je svaki harmonijski vektor nula-paralelno vektorsko polje.

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ON THE EXISTENCE OF A SELF-RECURRENT SW-ON

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Abstract

A recent paper treats the relations between the basic objects P, Q, π, $\tilde{\pi}$, $\hat{\pi}$ of a self-recurrent SW-On. There is a number of such relations in a Riemannian space if it is to serve as a basis for a self-recurrent SW-On.

Introduction

A space of the general regular connection, defined and investigated by T. Otsuki ([2], [3], [4], [5]) is a differentiable manifold μ, supplied by the basic covariant differentiation, given by

$$(0.1)\qquad T^i_{jk|h} = \frac{\partial T^i_{jk}}{\partial x^h} + {}'\Gamma^i_{sh} T^s_{jk} - {}''\Gamma^s_{jh} T^i_{sk} - {}''\Gamma^s_{kh} T^i_{js}$$

and by the general covariant differentiation, given by

$$(0.2)\qquad T^i_{jk,h} = P^i_a T^a_{bc|h} P^b_j P^c_k$$

where (P^i_j) denotes a C^∞ field of a nonsingular tensor of type (1,1). (Q^i_j) denotes the inverse of the tensor (P^i_j). The coefficients of two classical affine connections $({}'\Gamma^i_{jk})$ and $({}''\Gamma^i_{jk})$ (usually called the contravariant and covariant part of the regular general connection) are connected mutually by the fact

$$(0.3)\qquad Q^i_{j|k} = 0 .$$

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If the space of the regular general connection is a metric space, provided with a metric tensor (g_{ij}), we define an SW-On in the following way:

a) $g_{ij|k} = \gamma_k m_{ij}$ (γ_k is a vector field and m_{ij} is a symmetric tensor field)

b) $'\Gamma^i_{jk}$ is symmetric

c) $P_{jk} = g_{ja} P^a_k$ is symmetric

According to the formulae expressing $'\Gamma^i_{jk}$ and $''\Gamma^i_{jk}$ ([6], [7]) we have always paid special attention to the connections $'\overset{m}{\Gamma}{}^i_{jk}$ and $''\overset{m}{\Gamma}{}^i_{jk}$, which correspond to $\gamma_k=0$. $\{''\overset{m}{\Gamma}{}^i_{jk}\}$ is an object of a nonsymmetric metric affine connection and $'\overset{m}{\Gamma}{}^i_{jk}$ is its contravariant mate, which is not a metric connection.

Since we have given a metric tensor (g_{ij}) and a differentiable manifold M, then we have given a Riemannian geometry. We have named it the adjoint Riemannian space to SW-On.

The SW-On is self-recurrent if its fundamental (in some sense) tensor (P^i_j) is recurrent to the unit tensor in the adjoint Riemannian space

(0.4) $$\nabla_k P^i_j = \pi_k \delta^i_j$$

and if the vector $\tilde{\pi}_i = Q^a_i \pi_a$ satisfies the concircularity condition

(0.5) $$\nabla_k \tilde{\pi}_h = \tilde{\pi}_k \tilde{\pi}_h + \rho g_{kh} \qquad (\rho \text{ is a scalar function})$$

Then the corresponding ($''\overset{m}{\Gamma}{}^i_{jk}$) is a concircularly semi-symmetric metric connection

(0.6) $$''\overset{m}{\Gamma}{}^i_{jk} = \left\{ {i \atop jk} \right\} + \tilde{\pi}_j \delta^i_k - \tilde{\pi}^i g_{jk}$$

and, besides

(0.7) $$\nabla_k \pi_h = \tilde{\pi}_k \pi_h + \pi_k \tilde{\pi}_h + \rho P_{kh}$$

In [8], we proved that concircular and projective curvature tensor of a self-recurrent SW-On (that is, of the connection $''\overset{m}{\Gamma}$) are equal to the same tensors in the adjoint Riemannian space. In [9], we gave some properties of its Ricci and scalar curvature. Now, we want to give some conditions which the basic objects of a SW-On must satisfy.

1. Rieman-Cristoffel tensors, Ricci tensors and scalar curvatures

Taking into account (0.5) and (0.6), we can easily get

$$(1.1) \qquad ''\overset{m}{R}_{ijkh} = R_{ijkh} + (2\rho + \omega)(g_{ik}g_{jh} - g_{ih}g_{jk}),$$

where $''\overset{m}{R}_{ijkh}$ denote the Rieman-Cristoffel tensor of $''\overset{m}{\Gamma}$, R_{ijkh} the same tensor of the adjoint Riemannian space, ρ is the scalar function from (0.5) and ω denotes the length of the vector $(\tilde{\pi}_i)$, which is also a scalar function.

Contracting (2.1) by g^{ih}, we get

$$(2.2) \qquad ''\overset{m}{R}_{jk} = R_{jk} - (n-1)(2\rho+\omega)g_{jk}$$

which gives the relation between the Ricci tensor of $''\overset{m}{\Gamma}$ and the same tensor in the adjoint Riemannian space.

If we contract (2.2) now by g^{jk}, we are getting

$$(1.3) \qquad ''\overset{m}{R} = R - n(n-1)(2\rho+\omega)$$

which gives us the relation between the scalar curvature of $''\overset{m}{\Gamma}$ and the scalar curvature of the adjoint Riemannian space.

2. Integrability conditions for the existence of a self-recurrent SW-On

Let us now consider the scalar function ω. Its partial derivative can be expressed in this way

$$(2.1) \qquad \omega_k = \frac{\partial \omega}{\partial x^k} = \nabla_k \omega = 2(\rho+\omega)\tilde{\pi}_k$$

taking into account (0.5) and (0.6). The system of partial differential equations (2.1) is integrable if the following integrability condition

$$\frac{\partial^2 \omega}{\partial x^k \partial x^h} = \frac{\partial^2 \omega}{\partial x^h \partial x^k}$$

is satisfied. Differentiating (2.1) one more time, we get

(2.2) $$\omega_{kh} = \frac{\partial^2 a}{\partial x^k \partial x^h} = 2(\rho_h + 2(\rho+\omega)\tilde{\pi}_h)\tilde{\pi}_k + 2(\rho+\omega)(\tilde{\pi}_k\tilde{\pi}_h + \rho g_{kh} + \left\{ {}^{s}_{kh} \right\}\tilde{\pi}_s) .$$

Alternating the indices k and h, it follows from (2.2) and the integrability condition

(2.3) $$\rho_h\tilde{\pi}_k = \rho_k\tilde{\pi}_h ,$$

which is itself an integrability condition. ρ_k stands for $\frac{\partial_s}{\partial x^k}$. Besides, we can see that

(2.4) $$\frac{\partial\tilde{\pi}_k}{\partial x^h} = \tilde{\pi}_k\tilde{\pi}_h + \rho g_{hk} + \left\{ {}^{s}_{hk} \right\}\tilde{\pi}_s = \frac{\partial\tilde{\pi}_h}{\partial x^k}$$

Now, we can formulate

Lemma 1. *The necessary condition for a self-recurrent SW-On to exist is* (2.3). *Expect for this, the vector* $(\tilde{\pi}_k)$ *is locally a gradient.*

Now, we are going to examine the integrability condition for the scalar function μ. μ denotes the scalar product of vectors (π_i) and $(\tilde{\pi}_i)$.

(2.5) $$\mu_k = \frac{\partial\mu}{\partial x^k} = \nabla_k\mu = \nabla_k(\tilde{\pi}^s\pi_s) = 2\mu\tilde{\pi}_k + (2\rho+\omega)\pi_k .$$

The partial derivative of the second order is expressed in this way

(2.6) $$\frac{\partial^2\mu}{\partial x^k\partial x^h} = 2\mu_h\tilde{\pi}_k + 2\mu(\tilde{\pi}_k\tilde{\pi}_h + \rho g_{kh} + \left\{ {}^{s}_{kh} \right\}\tilde{\pi}_s) +$$
$$+ 2(\rho_h + \omega_h)\pi_k + (2\rho+\omega)(\tilde{\pi}_k\pi_h + \pi_k\tilde{\pi}_h + \rho P_{kh} + \left\{ {}^{s}_{kh} \right\}\tilde{\pi}_s) .$$

After alternating the indices k and h, we get the integrability condition:

(2.7) $$(\rho\tilde{\pi}_k + \rho_k)\pi_h = (\rho\tilde{\pi}_h + \rho_h)\pi_k$$

and

Lemma 2. *The necessary condition for a self-recurrent SW-On to exist is* (2.7).

Let us denote the scalar product $\pi_s\pi^s$ by $\mathcal{H}$. It is a scalar function. Differentiating $\mathcal{H}$ in the same manner as we have already done with functions ω and μ, we get

(2.8) $$\mathcal{H}_k = 2(\mathcal{H}\tilde{\pi}_k + \mu\pi_k + \rho\hat{\pi}_k),$$

where $\hat{\pi}_k$ denotes $\pi_a P^a_k$. For such a vector, there hold

(2.9) $$\nabla_h \hat{\pi}_k = 2\pi_k\pi_h + \tilde{\pi}_h\hat{\pi}_k + \rho \overset{2}{P}_{kh}$$

$(\overset{2}{P}_{kh} = P_{ka}P^a_h)$

Differentiating (2.8) one more time, we get

(2.10) $$\frac{\partial^2\mathcal{H}}{\partial x^h \partial x^k} = 2\Big[\mathcal{H}_h\tilde{\pi}_k + \mathcal{H}(\tilde{\pi}_h\tilde{\pi}_k + \rho g_{kh} + \left\{{s \atop kh}\right\}\tilde{\pi}_s + \\ + \mu_h\pi_k + \mu(\tilde{\pi}_h\pi_k + \tilde{\pi}_k\pi_h + \rho P_{kh} + \left\{{s \atop kh}\right\}\pi_s) + \\ + \rho_h\tilde{\pi}_k + \rho(2\pi_k\pi_h + \tilde{\pi}_h\hat{\pi}_k + \rho\overset{2}{P}_{kh})\Big]$$

Alternating the indices k and h in (2.10) and using the general integrability condition, we get

(2.11) $$2\rho\hat{\pi}_h\tilde{\pi}_k + \rho_h\hat{\pi}_k = 2\rho\hat{\pi}_k\tilde{\pi}_h + \rho_k\hat{\pi}_h$$

Then, there follows

Lemma 3. *The necessary condition for a self-recurrent SW-On to exist is* (2.11).

3. Other conditions

The other group of existence conditions can be obtained by the Ricci identities. For a covector field (v_m) and the connection $''\overset{m}{\Gamma}$, the Ricci identity is

(3.1) $$''\overset{m}{\nabla}_l\, ''\overset{m}{\nabla}_k v_h - ''\overset{m}{\nabla}_k\, ''\overset{m}{\nabla}_l v_h = ''\overset{m}{R}{}^s_{mlk} v_s + ''\overset{m}{T}{}^s_{lh}\, ''\overset{m}{\nabla}_s v_h .$$

Transforming (3.1) by formula (1.1), the Ricci identity for the Riemannian connection and by the fact that the connection $''\overset{m}{\Gamma}$ is semi-symmetric, we get

$$(3.2)\quad ''\overset{m}{\nabla}_l ''\overset{m}{\nabla}_k v_h - ''\overset{m}{\nabla}_k ''\overset{m}{\nabla}_l v_h = \nabla_l \nabla_k v_h - \nabla_k \nabla_l v_h + (2\rho+\omega)(v_l g_{kh} - v_k g_{lh}) + \tilde{\pi}_l ''\overset{m}{\nabla}_k v_h - \tilde{\pi}_k ''\overset{m}{\nabla}_l v_h .$$

We shall apply formula (3.2) to vectors $(\tilde{\pi}_h)$ and (π_h) first. For the vector $(\tilde{\pi}_h)$, we have from (0.5)

$$(3.3)\quad \nabla_l \nabla_k \tilde{\pi}_h - \nabla_k \nabla_l \tilde{\pi}_h = \rho \tilde{\pi}_k g_{lh} - \rho \tilde{\pi}_l g_{hk} + \rho_l g_{kh} - \rho_k g_{lh} ;$$

$$(3.4)\quad ''\overset{m}{\nabla}_k \tilde{\pi}_h = (\rho+\omega) g_{kh}$$

and

$$(3.5)\quad ''\overset{m}{\nabla}_l ''\overset{m}{\nabla}_k \tilde{\pi}_h - ''\overset{m}{\nabla}_k ''\overset{m}{\nabla}_l \tilde{\pi}_h = (\rho_l + \omega_l) g_{kh} - (\rho_k + \omega_k) g_{lh} .$$

Combining the results of (3.3), (3.4) and (3.5) with (3.2) and (2.1) we get an identity and no new conditions.

For the vector (π_h), it follows from (0.7) that

$$(3.6)\quad \nabla_l \nabla_k \pi_h - \nabla_k \nabla_l \pi_h = P_{lh}(\rho \tilde{\pi}_k - \rho_k) - P_{kh}(\rho \tilde{\pi}_l - \rho_l)$$

$$(3.7)\quad ''\overset{m}{\nabla}_k \pi_h = \tilde{\pi}_k \pi_h + \rho P_{kh} + \mu g_{kh}$$

and

$$(3.8)\quad ''\overset{m}{\nabla}_l ''\overset{m}{\nabla}_k \pi_h - ''\overset{m}{\nabla}_k ''\overset{m}{\nabla}_l \pi_h = (\mu \tilde{\pi}_k - \mu_k) g_{lh} - (\mu \tilde{\pi}_l - \mu_l) g_{kh}$$

Combining (3.6), (3.7) and (3.8) with (3.2), we get

$$(3.9)\quad \Big[2\mu\tilde{\pi}_k - \mu_k + (2\rho+\omega)\pi_k\Big] g_{lh} - \Big[2\tilde{\mu}\pi_l - \mu_l + (2\rho+\omega\pi_l)\Big] g_{kh} =$$

$$= \rho_l P_{kh} - \rho_k P_{lh} .$$

Applying (2.5) to (3.9), we get an identity again and no new conditions for the existence of a self-recurrent SW-On. The fact, that (3.5) and (3.9) are identities means that the structure of the self-recurrent SW-On is relatively compatible.

In the same fashion, we can investigate the fundamental (in some sense) tensors *P* and *Q*. The investigation of the tensor *Q* would be very simple because there hold

(3.10) $$''\overset{m}{\nabla}_k Q_{ij} = Q_{ij|k} = 0$$

and

(3.11) $$''\overset{m}{\nabla}_l\, ''\overset{m}{\nabla}_k Q_{ij} - ''\overset{m}{\nabla}_k\, ''\overset{m}{\nabla}_l Q_{ij} = ''\overset{m}{R}{}^s_{ilk} Q_{sj} + ''\overset{m}{R}{}^s_{jlk} Q_{ij} = 0 .$$

Besides, from (0.4), we can easily get

(3.12) $$\nabla_k Q_{ij} = - \pi_k Q_{ia} Q^a_j$$

and, using (0.7)

(3.13) $$\nabla_l \nabla_k Q_{ij} - \nabla_k \nabla_l Q_{ij} = R^s_{ilk} Q_{sj} + R^s_{jlk} Q_{is} = 0 .$$

Combining (3.13) with (3.11) and (1.1), we get

(3.14) $$(2\rho+\omega)(g_{ik} Q_{lj} - g_{il} Q_{kj} + g_{jk} Q_{il} - g_{jl} Q_{ik}) = 0$$

Then, we can state

Lemma 4. (3.14) *is an existence condition for a self-recurrent SW-On.*

For the tensor *P*, there hold, from (0.4),

(3.15) $$\nabla_l \nabla_k P_{ij} = g_{ij} \nabla_l \pi_k$$

and, consequently, by (0.7)

(3.16) $$\nabla_l \nabla_k \ddot{\;}_{ij} - \nabla_k \nabla_l P_{ij} = R^s_{ilk} P_{sj} + R^s_{jlk} P_{is} = 0$$

By (0.4) and (0.6), we have

$$''\overset{m}{\nabla}_k P_{ij} = \pi_k g_{ij} + \pi_i g_{jk} + \pi_j g_{ik} - \tilde{\pi}_i P_{kj} - \tilde{\pi}_j P_{ik}$$

and

(3.17) $$''\overset{m}{\nabla}_l\, ''\overset{m}{\nabla}_k P_{ij} - ''\overset{m}{\nabla}_k\, ''\overset{m}{\nabla}_l P_{ij} = g_{jk} \tilde{\pi}_l \pi_i - g_{jl} \tilde{\pi}_k \pi_i + g_{ik} \tilde{\pi}_l \pi_j - g_{il} \tilde{\pi}_k \pi_j + \\ + (2\rho+\omega)(P_{il} g_{jk} - P_{ki} g_{jl} + P_{lj} g_{ik} - P_{kj} g_{il}) .$$

But,

$$''\overset{m}{\nabla}_l\, ''\overset{m}{\nabla}_k P_{ij} - ''\overset{m}{\nabla}_k\, ''\overset{m}{\nabla}_l P_{ij} = (2\rho+\omega)(P_{lj} g_{ik} - P_{kj} g_{il} + P_{il} g_{jk} - P_{ik} g_{jl}) \\ + (\tilde{\pi}_l \delta^s_k - \tilde{\pi}_k \delta^s_l)\, ''\overset{m}{\nabla}_s P_{ij}$$

Then,

$$g_{ij}\tilde{\pi}_l\pi_l - g_{jl}\tilde{\pi}_k\pi_l + g_{ik}\tilde{\pi}_l\pi_j - g_{il}\tilde{\pi}_k\pi_j = (\tilde{\pi}_l\delta^s_k - \tilde{\pi}_k\delta^s_l)''\overset{m}{\nabla}_s P_{ij}$$

and, finally

$$(3.18) \qquad \tilde{\pi}_l\pi_k g_{ij} - \tilde{\pi}_l\tilde{\pi}_i P_{ik} - \tilde{\pi}_l\tilde{\pi}_j P_{ik} = \tilde{\pi}_k\pi_l g_{ij} - \tilde{\pi}_k\tilde{\pi}_l P_{ij} - \tilde{\pi}_k\tilde{\pi}_j P_{il} .$$

Then, we can state

Lemma 5. (3.18) *is an existence condition for a self-recurrent SW-On.*

4. Some special cases for functions ω, $\mathcal{H}$ and μ

Let us consider a possible case: that (π_h) is an eigen vector of the isomorphism Q, with an eigen value λ (λ is a C^r function of several variables). Then, $\tilde{\pi}_h=\lambda\pi_h$ and

$$(4.1) \qquad \nabla_k\tilde{\pi}_h = \lambda^2\pi_k\pi_h + \rho g_{kh}$$

$$(4.2) \qquad \nabla_k\tilde{\pi}_h = \nabla_k(\lambda\pi_h) = \lambda(2\lambda\pi_k\pi_h + \rho p_{kh}) + \lambda_k\pi_h ,$$

where λ_k denotes the k-th partial derivative of the function λ.

If (π_h) is an eigen vector of the isomorphism Q, it is an eigen vector of the isomorphism P, with an eigen value $\frac{1}{\lambda}$.

Combining (4.2) with (4.1) and transvecting by π^h, we get

$$(4.3) \qquad \lambda_k = -\lambda^2\pi_k .$$

Now, we have the following relations for the function $\mathcal{H}$, μ and λ

$$(4.4) \qquad \mu = \lambda\mathcal{H}, \quad \omega = \lambda^2\mathcal{H} .$$

From (4.3) we can see that λ is a constant function if and only if λ=0 globally, which can never happen,

Then, there follows:

$$(4.5) \qquad \mathcal{H}_k = 2(\mathcal{H}\tilde{\pi}_k + \mu\pi_k + \rho\tilde{\pi}_k) = (4\mathcal{H}\lambda + \frac{2\rho}{\lambda})\pi_k$$

$$(4.6) \qquad \mu_k = 2\mu\tilde{\pi}_k + (2\rho + \omega)\pi_k = (3\lambda^2 + 2\rho)\pi_k$$

$$(4.7) \qquad \omega_k = 2(\rho + \omega)\tilde{\pi}_k = 2\lambda(\rho + \lambda^2\mathcal{H})\pi_k .$$

Now, we have the next

Proposition 1. *If the vector* (π_k) *is an eigen vector of the isomorphism Q, with an eigen value* λ, *then*

(a) the vector (π_k) *is of constant length if and only if* $\rho=-2\mathcal{H}\lambda^2$

($\mathcal{H}$ *is the square of the length)*

(b) the vector $(\tilde{\pi}_k)$ *is of constant length if and only if* $\rho=-\lambda^2\mathcal{H}$

(c) the scalar product of vectors (π_k) *and* $(\tilde{\pi}_k)$ *is constant if*

$$\rho=-\frac{3\mathcal{H}\lambda^2}{2} \quad \text{or} \quad \rho=C\mu.$$

But, since the vector (π_k) and $(\tilde{\pi}_k)$ are collinear, their scalar product can be constant if and only if the lengths are constant. Then, the conjuction of (a) and (b) is equivalent to (c). But it is possible if and only if $\lambda^2\mathcal{H}=0$. Since λ is never zero, then $\mathcal{H}=0$ and there follows:

Theorem 1. *If the vector* (π_i) *is an eigen vector of the isomorphism Q, with a* C^r *eigen value,* λ *then*

(i) λ *is never a constant*

(ii) (π_i) *and* $(\tilde{\pi}_i)$ *are of constant length and their scalar product is constant if and only they are isotropic.*

5. Some considerations of an indefinite metric form of underlying Riemannian geometry

Surely, the vector (π) should not be an eigen vector of the isomorphism Q. In a more common case, vectors (π_k) and $(\tilde{\pi}_k)$ are linearly independent and form a 2-plane Π. Now, if we want the vector $(\tilde{\pi}_h)$ to be of constant length, then

$$\omega_k = (2\rho+\omega)\tilde{\pi}_k = 0$$

and, consequently,

$$\rho = -\omega . \tag{5.1}$$

Then, applying (5.1) to (1.1), we get

$$''\overset{m}{R}{}^i_{jkh} = R^i_{jkh} - c(\delta^i_k g_{jk} - \delta^i_h g_{jk}), \tag{5.2}$$

where c denotes ω, for the sake of its invariability.

Then, taking into account formula (1.3), we have

Lemma 6. *If the length of the vector* (π_h) *is constant, then the difference between the scalar curvatures of* $''\overset{m}{\Gamma}$ *in a self-recurrent SW-On and its adjoint Riemannian space is constant.*

An even more interesting result is gained for the case μ =const. As we have supposed, (π_k) and $(\tilde{\pi}_k)$ are linearly independent and consequently, (μ_k) is an element of the 2 plane Π. Then, it is evident that if μ =const, then $\mu=0$ and $2\rho+\omega=0$.

Lemma 7. *If the vectors* (π_k) *and* $(\tilde{\pi}_k)$ *are linearly independent, their scalar product is a constant if and only if they are orthogonal.*

But, if they are orthogonal, $2\rho+\omega=0$ too. Then, we can state

Lemma 8. *If the vectors* (π_k) *and* $(\tilde{\pi}_k)$ *are orthogonal, the curvature tensor of* $''\overset{m}{\Gamma}$ *in a self-recurrent SW-On is equal to the curvature tensor of the adjoint Riemannian space.*

For the tensor $'\overset{m}{R}$ and $''\overset{m}{R}$, there holds the relation

$$''\overset{m}{R}{}^{s}_{pkl}\, Q^{i}_{s}\, P^{p}_{j} = {}'\overset{m}{R}{}^{i}_{jkl} \qquad ([1]) \,. \tag{5.3}$$

But, in the case of the orthogonality of the vectors (π_k) and $(\tilde{\pi}_k)$, $''R=R$ and, from the Ricci identity for the tensor Q (3.13), we have

$$R^{s}_{pkl}\, Q^{i}_{s} = R^{si}_{lk}\, Q_{sp}$$

and, transvecting by P^{p}_{j},

$$R^{s}_{pkl}\, Q^{i}_{s}\, P^{p}_{j} = R^{si}_{lk}\, Q_{sp}\, P^{p}_{j}$$

i.e.

$$'R^{i}_{jkl} = R^{si}_{lk}\, g_{sj} = R^{i}_{jlk} \,. \tag{5.4}$$

We can see that the curvature tensor $'\overset{m}{R}$ is not equal to the curvature tensor of the adjoint Riemannian space, but lowering the superscript i, we get the relation

$$'\overset{m}{R}_{ijkl} = R_{jilk} = R_{ijkl} \tag{5.5}$$

Now, we can state

Theorem 2. *If the vectors* (π_k) *and* $(\tilde{\pi}_k)$ *are orthogonal, the Rieman-Cristoffel tensors* $'\overset{m}{R}$ *and* $''\overset{m}{R}$ *are both equal to the Rieman-Cristoffel tensor of the adjoint Riemannian space. The same relation holds for their Ricci tensor and scalar curvatures.*

In that case, we call R_{ijkl} a Rieman-Cristoffel tensor of a self-recurrent SW-On, R_{jk} a Ricci tensor of a self-recurrent SW-On and $R=R_{jk}g^{jk}$ a scalar curvature of a self-recurrent SW-On.

We have a similar situation for $\omega=0$ ($\tilde{\pi}_h$ an isotropic vector). Then, from (2.1)

$$\omega_k = 2(\rho+\omega)\tilde{\pi}_k = 0$$

and $\rho=\omega=0$.

Proposition 2. *If* $\tilde{\pi}_h$ *is an isotropic vector field, Lemma 8. and Theorem 2. hold for such a self-recurrent SW-On. Moreover, such a vector is a harmonic vector field in the adjoint Riemannian space (pseudo-Riemannian space) and it is parallel with respect to connection* $''\overset{m}{\Gamma}$.

For the vector $\hat{\pi}_k = \pi_a P^a_k$, *we have two cases: it may be an element of* Π, *if* Π *is an invariant 2-plane or* π_k, $\tilde{\pi}_k$ *and* $\hat{\pi}_k$ *may be linearly independent.*

If these vectors are linearly independent, then (π_k) *can be of a constant length if and only if*

(a) π_k *is an isotropic vector field*

(b) π_k *and* $\tilde{\pi}_k$ *are orthogonal*

(c) $\rho = -\dfrac{\omega}{2} = 0$

(from (2.1), (2.5) *and* (2.8)).

Proposition 3. *If* Π *is not an invariant 2-plane of the isomorphism Q and if the vector* (π_k) *is of a constant length, then*

(1) π_k *and* $\tilde{\pi}_k$ *are orthogonal*

(2) π_k *is an isotropic vector field and the adjoint Riemannian space is a pseudo-Riemannian space*

(3) $\tilde{\pi}_k$ *is an isotropic vector field*

(4) *the Riemann-Cristoffel tensor of the adjoint pseudo-Riemannian space is equal to both* $''\overset{m}{R}_{ijkh}$ *and* $'\overset{e}{R}_{ijkh}$.

If the 2-plane Π is an invariant plane for the isomorphism Q, then the fact that π_k is of a constant length does not cause its isotropy.

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Rezime

O POSTOJANJU SAMOREKURENTNOG SW-On

U radu su ispitani neki neophodni uslovi koje moraju zadovoljavati osnovni objekti P, Q, $\tilde{\pi}$, $\hat{\pi}$, da bi se nad datim prostorom kao nad baznim mogao konstruisati SW-On. Takođe, ispitani su i neki slučajevi kada pridruženi Rimanov prostor ima idefinitnu metriku.

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COUNTING, GENERATION AND RECOGNITION OF S-SEQUENCES

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Abstract

S-words (also called *S*-sequences) are of importance in the theory of n-ary structures. In this paper it is proved that the number of different *S*-words of the length n is the $(n-1)$-th Catalan number. A simple way of generating all the different *S*-words of the length n is given. Also an algorithm is given for the recognition of *S*-words. The complexity of this algorithm is $O(n)$, where n is the length of the word.

1. Some definitions and notation

The following scheme will be called the *S*-scheme:

0
11
221, 122
3321, 2331, 2222, 2222, 1332, 1233

It consists of the sequences (words) of non-negative integers generated in the following way. Let a be a word in the i-th row of the *S*-scheme. Replacing a number k of the word a by the subword $(k+1)(k+1)$ (we denote such a replacement by $k \to (k+1)(k+1)$), we obtain a word in the $(i+1)$-th row of the *S*-scheme. Since the only word of the first row is 0, we conclude that all

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the words of the *i*-th row have the same length *i*. In such a way each word of the *i*-th row produces *i* words of the (*i*+1)-th row. So, the number of words in the *i*-th row is *i*! They are not all different, however. For example, the word 2222 appears twice in the fourth row.

We call the words of the *S*-scheme *S*-words or *S*-sequences.

Let α be an integer. By replacing each number *k* in the *S*-scheme with $k+\alpha$, we obtain the so-called $(S+\alpha)$-scheme. For example,

2
33
443, 344
5543, 4553, 4444, 4444, 3554, 3455
.
.
.

is an $(S+2)$-scheme.

A word from the *n*-th row of the *S*-scheme has a length *n*. We also say that it is an S-(n) word.

In [1] some properties of *S*-words are examined. It is proved, for example, that in the *n*-th row of the *S*-scheme there are not two words $a = a_1a_2\ldots a_n$ and $b = b_1b_2\ldots b_n$ such that $a_i < b_i$ for each $i = 1,2,\ldots n$. The problem is posed by Cupona. The investigation of *S*-words was inspired by their applications in the theory of *n*-ary structures.

We define the weigth $w(a)$ of an *S*-word $a=a_1a_2\ldots a_n$ in the following way:

$$w(a) = \sum_{i=1}^{n} a_i \ .$$

In [1] it is also proved that for an S-(n) word the maximal weigth is

$$\binom{n+1}{2} - 1$$

and the minimal weigth is

$$2(n - 2^{[\log n]}) + n\,[\log n],$$

where $\log n = \log_2 n$ and $[x]$ is the greatest integer $\le x$.

We shall also use the following notation:

$\|a\|$ - the length of the word *a*;

x_k^a - the number of appearances of the number *k* in the word *a*;

y_k^a - the number of applications of the replacement $k\text{-}1 \longrightarrow kk$ during the process of producing the word *a*.

2. Counting of S-(n) words

First, we shall prove an auxiliary statement.

Theorem 1. *If a and b are two S-words such that* $\|a\| < \|b\|$, *then there is a non-negative integer k such that* $x_k^a > x_k^b$.

Proof. For an arbitrary *S*-word *c* we have:

$$\|c\| = 1 + \sum_{i\geq 1} y_i^c$$

and

(1) $$x_j^c = 2y_j^c - y_{j+1}^c ,$$

for $j=1,2,..$

If $\|a\| < \|b\|$, then

(2) $$\sum_{i\geq 1} y_i^a < \sum_{i\geq 1} y_i^b .$$

Suppose that, for each *i*, $y_i^a \geq y_i^b$.

Then,

$$\sum_{i\geq 1} y_i^a \geq \sum_{i\geq 1} y_i^b ,$$

which is in contradiction with (2).

Hence, it follows that for some $k \geq 0$,

(3) $$y_{k+1}^a < y_{k+1}^b .$$

If $y_1^a < y_1^b$, then $x_0^a > x_0^b$. Otherwise, take the least such integer $k \geq 1$. Then,

(4) $$y_k^a \geq y_k^b .$$

Now, from (1) it follows that

$$x_k^a = 2y_k^a - y_{k+1}^a , \quad x_k^b = 2y_k^b - y_{k+1}^b ,$$

and taking into account (3) and (4) we obtain

$$x_k^a > x_k^b . \quad \square$$

Corollary. *If* $a = a_1 a_2 \dots a_n$ *is an S-word, then for any* $m = 1,2,\dots,n-1$, $a' = a_1 a_2 \dots a_m$ *is not an S-word.*

Theorem 2. *The number of different S-(n)words is*

$$\frac{1}{n}\binom{2n-2}{n-1} .$$

Proof. Denote the number of different S-(n)words by $f(n)$. It is clear that at the same time $f(n)$ is the number of different $(S+\alpha)$-(n) words, for any integer α .

Let a be an arbitrary S-(n)word. This word can be represented as a concatenation of some two $(S+1)$-words b and c, i.e. $a = bc$. From the Corollary of Theorem 1, it follows that such a representation is unique. On the other hand, it is clear that the concatenation of any two $(S+1)$-words is an S-word.

Hence, it follows that the number $f(n)$ satisfies the recurrence relation

$$f(n) = \sum_{k=1}^{n-1} f(k)\, f(n-k) .$$

Since obviously $f(1) = 1$, the solution of this recurrence relation is the very well known series of Catalan numbers (see e.g. [2]), i.e.

$$f(n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} . \tag{5}$$

$\square$

Now, we are going to determine the number of S-word of special types.

Theorem 3. *The number of different S-(n)words of maximal weigth is* 2^{n-1}, *for* $n \geq 2$.

Proof. An S-(n)word has the maximal weigth iff in the process of generation of this word we apply each replacement to the maximal number. This means that at the k-th step a number k-1 will be replaced be kk. In such a word, the maximal number always appears exactly twice in two adjacent places. Hence, there are two possibilities at each step, except at the first one when we have only one possibility: 0 → 11. It follows that in the n-th row of the S-scheme we obtain 2^{n-2} words of maximal weigth. They are all different. Indeed, it is obvious for n=2. Suppose that it is true for n=k. In each S-(k) word the maximal element appears in exactly two adjacent places. Now, any two S-(k+1) words obtained from two different S-(k) words of maximal weigth are obviously different. On the other hand, two S-(k+1) words of maximal weigth obtained from the same S-(k) word of maximal weigth differ in places in which the first maximal numbers appear. □

Theorem 4. *The number of different S-(n) words of minimal weigth is*

$$\binom{2^{[\log n]}}{n - 2^{[\log n]}},$$

for $n \geq 1$.

Proof. It is obvious that for $n=2^k$ ($k \in N$), the only S-(n) word of minimal weigth is of the form

$$kk\ldots k. \tag{6}$$

So, the number of such words is

$$1 = \binom{2^k}{2^k - 2^k} = \binom{2^{[\log n]}}{n - 2^{[\log n]}}.$$

Now, suppose that $n = 2^k + r$, where $1 \leq r < 2^k$. Then, any S-(n) word of minimal weigth has $2^k - r$ letters (numbers) k and $2r = 2(n-2^k)$ letters (k+1). Such a word is obtained from word (6) using $r = n-2^k$ replacements of the form $k \to (k+1)(k+1)$. A choice of r letters k to be replaced can be made in

$$\binom{2^k}{r} = \binom{2^k}{n - 2^k} = \binom{2^{[\log n]}}{n - 2^{[\log n]}}$$

ways. Hence follows the statement. □

3. Recognition of S-words

The series 1, 1, 2, 5, 14, 42, 132,... i.e. $C_n = \frac{1}{n+1}\binom{2n}{n}$, for n=0,1,2,... occurs very often in counting problems. Some form of Catalan numbers appears every time we find a recurrence relationship of type (5). This series appears also as the solution of the following combinatorial problem [2].

We define an C-sequence as a sequence of integers $c_1 c_2 \ldots c_n$, such that

$$1 \le c_1 \le c_2 \le \ldots \le c_n \tag{7}$$

and

$$c_1 \le 1,\ c_2 \le 2, \ldots,\ c_n \le n\ . \tag{8}$$

It is very well known that the number of C-sequences of length n is

$$C_n = \frac{1}{n+1}\binom{2n}{n}\ .$$

Now, we are going to establish a bijection between the set of all S-$(n+1)$ words and the set of all the C-sequences of length n.

Let $c = c_1 c_2 \ldots c_n$ be a C-sequence. Then, starting form the S-word 0, we can generate an S-word $a = a_1 a_2 \ldots a_{n+1}$ in n steps as follows. The i-th step consists in replacing the c_i-th letter of the S-word. Condition (8) guaranties the possibility of correspondig replacements.

Example. The S-word corresponding to the C-sequence c = 113346 is generated in the following way:

$$\begin{array}{l} \underline{0} \\ \underline{1}1 \\ 22\underline{1} \\ 22\underline{2}2 \\ 223\underline{3}2 \\ 22344\underline{2} \\ 2234433\ . \end{array}$$

In the above example we first replace the first letter 0 by 11 (c_1=1), then the first letter 1 of word 11 by 22 (c_2=1), in the third step we replace the third letter 1 of word 221 by 22 obtaining the S-word 2222 etc.

The letters of a *C*-word can be considered as the sequence of instruction for the generation of the corresponding *S*-word. Instruction c_i means that the *i*-th replacement is applied to the *i*-th letter of the *S*-word.

We say that in this way an *S*-word is generated *from left to right.*

It is clear that different *S*-words correspond to different *C*-sequences. Since the number of different *S*-(n+1) words is the same (C_n) as the number of different *C*-sequences of length *n*, it follows that each *S*-word can be generated from left to right uniquely.

This unique way of generation enables us to consturct an algorithm for recognizing any *S*-(n) word in at most n-1 steps.

The recognition algorithm for S-words

Let $a=a_1a_2...a_n$ be a sequence of non-negative integers ($n \geq 2$). Starting from the *S*-word 0, we first apply a_1 replacements of the first letter. In this way we obtain an *S*-word $a'=a_1a'_2...a'_{a_1+1}$.

If $a'=a$, we conclude that *a* is an *S*-word. If $a_1+1 > n$ or $a_1+1 = n$ and $a' \neq a$, then *a* is not an *S*-word.

If $n > a_1+1$, we look for the first *i* such that $a_i \neq a'_i$. Such an *i* exists according to the Corollary of Theorem 1.

Now, if $a'_i > a_i$, this means that *a* is not an *S*-word. If $a'_i < a_i$, we apply $a_i - a'_i$ times the replacement of the *i*-th letter. In this way, we obtain a new *S*-word

$$a''= a_1a_2...a_ia''_{i+1}...a''_{a_1+a_i-a'_i} .$$

If, continuing in this way, after n-1 replacements we obtain the word a, this means that a is an *S*-word. Otherwise, after at most n-1 steps, we obtain a word $b= b_1b_2...b_m$, such that either

$m < n$ and *b* is a prefix of a

or

for some $j \in \{1,2,...,m\}$, $a_i = b_i$ if $i < j$ and $a_j < b_j$.

In that case we conclude that *a* is not an *S*-word.

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Rezime

PREBROJANJE, GENERISANJE I PREPOZNAVANJE S-REČI

S-reči (*S*-nizovi) nalaze primenu u teoriji n-arnih struktura. U ovom radu dokazano je da je broj različitih *S*-reči dužine *n* jednak (n-1)-om broju Katalana. Dat je jedan postupak za generisanje svih različitih *S*-reči dužine *n*. Dat je, takođe, jedan algoritam složenosti $O(n)$ za prepoznavanje *S*-reči dužine *n*.

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ON UNAVOIDABLE SUBGRAPHS OF STRONG TOURNAMENTS

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Abstract

If $C(n,i)$ denotes a simple n-cycle $v_1 \to v_2 \to \dots \to v_n \to v_1$ with an additional arc $v_j v_{j+i-1}$ for some $j \in \{1,2,\dots,n\}$ and $3 \le i \le n-1$ it is proved that every strong n-tournament T_n contains copies of $C(n, \lceil (n+2)/2 \rceil)$ and $C(n,n-2)$ for each n $(n \ge 4)$.

The terminology and notation is that of [1] except as noted.

T_n denotes an arbitrary n-tournament. $C(n,i)$ denotes a simple cycle $v_1 \to v_2 \to \dots \to v_n \to v_1$ with an additional arc $v_j v_{j+i-1}$ for some $j \in \{1,2,\dots,n\}$ and $3 \le i \le n-1$. $T_n - v$ denotes the subtournament of T_n obtained by deleting the vertex v and all incident arcs. For vertices u and v, $u \to v$ will be used to denote the phrase "u dominates v" and an arc from u to v in T_n. For two disjoint subsets A and B of $V(T_n)$, $A \to B$ denotes that every vertex of A dominates every vertex of B.

The following theorem will be of use.

Theorem 1. ([2]) *Every strong n-tournament T_n contains a copy of $C(n,i)$ for each n $(n \ge 4)$ and i, $3 \le i \le (n+2)/2$.*

This paper will present some results on digraphs $C(n,i)$ for those i's not covered by Theorem 1.

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Theorem 2. *Every strong n-tournament* T_n *contains a copy* $C(n, [n+2)/2])$ *for each* n $(n \geq 4)$.

Proof. For n even or $n \leq 5$ the statement follows immediately by Theorem 1. So, we may assume that $n = 2k + 1$ $(k \geq 3)$ and look for $C(2k+1,\ k+2)$ in strong $(2k+1)$-tournaments. We shall suppose that there is a strong $(2k+1)$-tournament T_{2k+1} containing no copy of $C(2k+1,\ k+2)$ and show that it leads to a contradiction.

Let T_n, $n=2k+1$ $(k \geq 3)$ be a strong tournament which contains no copy of $C(2k+1,\ k+2)$. Let v be a vertex of T_n such that the tournament $T_n - v$ is strong. (For the existence of such a vertex see [1] p.6.) By Theorem 1 the tournament $T_n - v$ contains a copy of $C(2k, k+1)$. Label vertices of $T_n - v$ so that

$$\underline{v_1} \rightarrow v_2 \rightarrow \ldots\ v_k \rightarrow \underline{v_{k+1}} \rightarrow \ldots \rightarrow v_{2k} \rightarrow v_1$$

is a copy of $C(2k,\ k+1)$. (The underlined pair of vertices denotes that the first one dominates the second.) The proof falls in to three cases.

Case 1. $v_1 \rightarrow v$. This implies

$$\{v_2, v_3, \ldots, v_{k+1}\} \rightarrow v. \tag{1}$$

Otherwise, if $v \rightarrow v_j$ for some $j \in \{2, 3, \ldots, k+1\}$, v can be inserted in the path $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{k+1}$ obtaining a copy of $C(2k+1,\ k+2)$ given by $\underline{v_1} \rightarrow \ldots \rightarrow v \rightarrow \ldots\ v_{k+1} \rightarrow \underline{v_{k+2}} \rightarrow \ldots \rightarrow v_k \rightarrow v_1$. So, (1) holds and as $T_n - v$ is strong there is at least one vertex v_i, $i \in \{k+2, k+3, \ldots, 2k\}$ such that $v \rightarrow v_i$. Let i_0 be the minimal of these i's i.e. $v_{i_0-1} \rightarrow v$ and $v \rightarrow v_{i_0}$. Then,

$$\underline{v_{i_0-k-1}} \rightarrow v_{i_0-k} \rightarrow \ldots \rightarrow v_{i_0-1} \rightarrow \underline{v} \rightarrow v_{i_0} \rightarrow v_{i_0+1} \rightarrow \ldots \rightarrow$$
$$\rightarrow v_{i_0-k-2} \rightarrow v_{i_0-k-1}$$

is a copy of $C(2k+1,\ k+2)$ as $1 \leq i_0 - k - 1 \leq k - 1$ and by (1) $v_{i_0-k-1} \rightarrow v$. Case 1 is settled.

Case 2. $v \rightarrow v_{k+1}$. Before discussing this case observe the following property of digraphs $C(n, i)$. Reversing all the arcs of $C(n, i)$ results in a digraph isomorphic to $C(n, i)$. Thus, after reversing all the arcs of T_n we get the tournament T_n which satysfies conditions of case 1 and contains a copy of $C(2k+1,\ k+2)$. Reversing now all the arcs of T_n we get the former tournament T_n and $C(2k+1, k+2)$ in it. In fact, case 2 is the dual of case 1.

Case 3. $v \longrightarrow v_1$ and $v_{k+1} \longrightarrow v$. We claim that now there exists $i_0, k+2 \le i_0 \le 2k$ such that

(2) $$v \longrightarrow \{v_{i_0+1}, v_{i_0+2}, \dots, v_{i_0+k}\}$$

(3) $$\{v_{i_0+k+1}, v_{i_0+k+2}, \dots, v_{i_0}\} \longrightarrow v$$

(All incides are reduced modulo $2k$.)

First notice if $v \longrightarrow v_j$ for some $j \in \{1,2,\dots,k\}$, then $v \longrightarrow$ $\longrightarrow \{v_1, v_2, \dots, v_j\}$. Indeed, if $v_m \longrightarrow v$ for some $m \in \{2,3,\dots,j-1\}$ then v can be inserted in the path $v_1 \longrightarrow v_2 \longrightarrow \dots v_j$ producing a copy of $C(2k+1,k+2)$ given by $\underline{v_1} \longrightarrow \dots \longrightarrow v \longrightarrow \dots \longrightarrow v_j \longrightarrow v_{j+1} \longrightarrow \dots \longrightarrow \underline{v_{k+1}} \longrightarrow v_{k+2} \longrightarrow \dots \longrightarrow v_{2k} \longrightarrow v_1$.

It follows that there exists j_0, $2 \le j_0 \le k$ such that

$$v \longrightarrow \{v_1, v_2, \dots, v_{j_0}\}$$
$$\{v_{j_0+1}, v_{j_0+2}, \dots, v_{k+1}\} \longrightarrow v.$$

Further, denote by i_0 the maximum of $k+1, k+2, \dots, 2k$, so that

$$\{v_{k+1}, v_{k+2}, \dots, v_{i_0}\} \longrightarrow v.$$

We shall show that $i_0 - j_0 \ge k$. (All computations will be done modulo $2k$.) Indeed, if $i_0 - j_0 < k$, then $\underline{v} \longrightarrow v_{i_0+1} \longrightarrow \dots \longrightarrow \underline{v_{i_0+k+1}} \longrightarrow v_{i_0+k+2} \longrightarrow \dots \longrightarrow v_{i_0} \longrightarrow v$ is a copy of $C(2k+1, k+2)$. (Notice that $v \longrightarrow v_{i_0+k+1}$ as $1 \le i_0+k+1 \le j_0$.) Thus, $i_0 - j_0 \ge k$.

Using the same argument we derive that there exists l_0, $k+2 \le l_0 \le 2k$ such that $j_0 - l_0 + \ge k$ and $v \longrightarrow \{v_1, v_{l_0+1}, \dots, v_{j_0}\}$. Since $|V(T_n - v)| = 2k$, it follows that

$$i_0 - j_0 \equiv j_0 - l_0 + 1 \equiv k \pmod{2k}$$

i.e. $l_0 = i_0 + 1$. Obviously, this implies (2) and (3).

Relabel now the vertex set of T_n as follows. The vertex v_{i_0+1} is labelled w_1, v_{i_0+2} is labelled $w_2, \dots, v_{i_0}$ is labelled w_{2k} and v is labelled w_{2k+1}. According to this labelling and having in mind (2) and (3), we find that

(4) $$w_1 \longrightarrow w_2 \longrightarrow \dots \longrightarrow w_{2k} \longrightarrow w_{2k+1} \longrightarrow w_1$$

is a Hamiltonian cycle of T_n where

(5)
$$w_{2k+1} \to \{w_1, w_2, \dots, w_k\}$$
$$\{w_{k+1}, w_{k+2}, \dots, w_{2k}\} \to w_{2k+1}$$

This implies that

(6)
$$w_i \to w_{i+k}$$

holds for each $i \in \{1,2,\dots,2k+1\}$. (All computations which follow will be done modulo $2k+1$.) For $i=k+1$ and $i=2k+1$ (6) follows by (5). Suppose that $w_{i+k} \to w_i$ for some $i \in \{1,2,\dots,k,k+2,k+3,\dots,2k\}$. Then, $\underline{w_{i+k}} \to w_{i+k+1} \to \dots \to \underline{w_i} \to w_{i+1} \to \dots \to w_{i+k-1} \to w_{i+k}$ is a copy of $C(2k+1,\ k+2)$ in T_n. This contradiction proves (6).

Consider now a copy of $C(2k,\ k+1)$ in $T_n - w_{2k}$ given by $\underline{w_{2k+1}} \to w_1 \to \to w_2 \to \dots \to \underline{w_k} \to w_{k+1} \to \dots \to w_{2k-1} \to w_{2k+1}$ ($w_{2k+1} \to w_k$ by (4).) As, by (4) and (6), $w_{2k} \to w_{2k+1}$, $w_k \to w_{2k}$, $w_{2k} \to w_{k-1}$ we conclude, using the same arguments as for (5), that

$$w_{2k} \to \{w_{2k+1}, w_1, w_2, \dots, w_{k-1}\} ;$$
$$\{w_k, w_{k+1}, \dots, w_{2k-1}\} \to w_{2k}$$

Continuing this procedure we obtain that

(7)
$$w_i \to \{w_{i+1}, w_{i+2}, \dots, w_{i+k}\}$$
$$\{w_{i+k+1}, w_{i+k+2}, \dots, w_{i-1}\} \to w_i$$

holds for each $i \in \{1,2,\dots,2k+1\}$. So, T_n is the regular tournament with an arc set given by (7). But such a tournament contains a copy of $C(2k+1, k+2)$ given by

$$\underline{w_1} \to w_3 \to w_5 \to \dots \to w_{2k+1} \to \underline{w_2} \to w_4 \to \dots w_{2k} \to w_1$$

This contradiction settles case 3 and completes the proof of the theorem.

Theorem 3. *Every strong n-tournament* T_n *contains a copy of* $C(n, n-2)$ *for each* $n (n \ge 4)$.

Proof. By induction on n. For $n \le 7$ the statement follows by Theorem 1. So, assume that $n \ge 8$. Let T_{n+1} $(n \ge 7)$ be an arbitrary strong $(n+1)$-tournament and let v be a vertex of T_{n+1} such that $T_{n+1} - v$ is strong too. By the induction hypothesis, $T_{n+1} - v$ contains a copy of $C(n, n-2)$ given by

(8) $$\underline{v_1} \to v_2 \to \dots \to \underline{v_{n-2}} \to v_{n-1} \to v_n \to v_1$$

Now we suppose that T_n does not contain a copy of $C(n+1,n-1)$ and show that it leads to a contradiction.

We shall consider three characteristic cases.

Case 1. $v_1 \to v$. Applying the same argument as in case 1 of Theorem 2 we obtain that

(9) $$\{v_2, v_3, \dots, v_{n-2}\} \to v$$

Also, since T_{n+1} is strong, some of the vertices v_{n-1} and v_n dominate v. Let i be the maximum of $n-1$ and n so that $v_i \to v$. Combining this with (7) and (8), we get a copy of $C(n+1,n-1)$ given by

$$\underline{v} \to v_{i+1} \to v_{i+2} \to \dots \to \underline{v_{i-2}} \to v_{i-1} \to v_i \to v$$

(Observe that $1 < i-2 \le n-2$ and $v \to v_{i-2}$) This contradiction settles case 1.

Case 2. $v \to v_{n-2}$. This case is the dual of case 1. Compare this with case 2 of Theorem 2.

Case 3. $v \to v_1$ and $v_{n-2} \to v$. As for the arcs connecting the vertex v with vetrices v_{n-1} and v_n, there are the four possible cases:

a) $v \to \{v_{n-1}, v_n\}$

b) $v \to v_{n-1}$, $v_n \to v$

c) $\{v_{n-1}, v_n\} \to v$

d) $v_{n-1} \to v$, $v \to v_n$

It is easy to see that (*a*) and (*c*) and also (*b*) and (*d*) are dual. So, we shall examine (a) and (*b*) only.

a) $v \to \{v_{n-1}, v_n\}$. First notice that $v_{n-4} \to v$. (If on the contrary, $v \to v_{n-4}$, then there is a copy of $C(n+1,n-1)$ given by $\underline{v} \to v_{n-1} \to v_n \to v_1 \to v_2 \to \dots \to \underline{v_{n-4}} \to v_{n-3} \to v_{n-2} \to v$.) This implies

(10) $$v_{n-3} \to v$$

Otherwise, we obtain a copy of $C(n+1,n-1)$ given by $\underline{v_1} \to v_2 \to \dots \to v_{n-4} \to v \to v_{n-3} \to \underline{v_{n-2}} \to v_{n-1} \to v_n \to v_1$. Furthermore,

(11) $$v_{n-1} \to v_2$$

In view of a copy of $C(n+1,n-1)$ given by $\underline{v_2} \to v_3 \to \dots \to v_{n-2} \to v \to$ $\to \underline{v_{n-1}} \to v_n \to v_1 \to v_2$.

Finally, using (10) and (11) we still obtain a copy of $C(n+1,n-1)$ given by

$$\underline{v_{n-1}} \to v_2 \to v_3 \to \dots v_{n-3} \to v \to v_n \to v_1 \to v_{n-2} \to v_{n-1} .$$

This contradiction settles (*a*).

(*b*) $v \to v_{n-1}$, $v_n \to v$. By the same argument as in (*a*) (10) and (11) holds. Furthermore,

(12) $$v_n \to v$$

because of the eventual copy of $C(n+1,n-1)$ given by $\underline{v_2} \to v_3 \to \dots \to \underline{v_n} \to$ $\to v \to v_1 \to v_2$. But, now, using (11) and (12) we obtain a copy of $C(n+1,n-1)$ given by

$$\underline{v_n} \to v_2 \to v_3 \to \dots \to v_{n-3} \to v \to \underline{v_1} \to v_{n-2} \to v_{n-1} \to v_n$$

contradicting the assumption.

The proof of the theorem is completed.

This paper and [2] partially confirm following, may be true,

Conjecture. *Every strong n-tournament* T_n *contains a copy of* $C(n,i)$ *for each* $n (n \geq 4)$ *and* $i (3 \leq i < n-1)$.

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2. V. Petrović: Some unavoidable subgraphs of strong tournamets (to appear)

Rezime

NEIZBEŽNI PODGRAFOVI JAKO POVEZANIH TURNIRA

U radu se pokazuje da svaki jako povezan turnir sa n čvorova sadrži kao podgraf $C(n,i)$, gde je $C(n,i)$ orjentisana kontura $v_1 \to v_2 \to \dots \to v_n \to v_1$ sa jednom dodatnim lukom $v_j v_{j+i-1}$, $j \in \{1,2,\dots,n\}$, $i = \lceil (n+2)/2 \rceil$, $i = n-2$, $n \geq 4$.

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THE FUZZY POWER OF ALGEBRAS

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Abstract

In the usual constructions of fuzzy algebraic structures, the lattice being the range of all the mappings is complete and often distributive. Complementedness (without distributivity) is rarely used in such constructions.

It is shown that a class of complemented lattices can be used to construct a lattice extension of an algebra and its fuzzy subalgebras.

1. Let L be a complete lattice with a zero (0) and unit (1) element, and $\mathcal{A}=(A,F)$ an arbitrary algebra. Let $A(L)$ be a collection of L-fuzzy sets on A (i.e. of all the mappings $\overline{X}\colon A \to L$), such that

1) $\overline{X}(a) \wedge \overline{X}(b) = 0$, for all $a,b \in A$, $a \neq b$;

2) $\bigvee_{a\in A} \overline{X}(a) = 1$.

Define the operations on $A(L)$ in the same way as for the Boolean power ([1]): if $f \in F_n \subseteq F$, and $\overline{X}_1,\dots,\overline{X}_n \in A(L)$, then

$f(\overline{X}_1,\dots,\overline{X}_n) = \overline{Y}$, and for $a \in A$

$\overline{Y}(a) = \bigvee (\overline{X}_1(a_1) \wedge \dots \wedge \overline{X}_n(a_n))$; $f(a_1,\dots,a_n = a)$.

AMS subject class: 03E72, 06C15

Keywords: fuzzy sets, lattices

We shall say that L *allows the fuzzy power* if for an arbitrary algebra $\mathcal{A}$,

(*i*) every $p \in L$ belongs to the image of some $\overline{X} \in A(L)$, and

(*ii*) 3) defines an operation on $A(L)$, i.e.

$$f(\overline{X}_1,\dots,\overline{X}_n) \in A(L), \text{ for all } \overline{X}_1,\dots,\overline{X}_n \in A(L) .$$

The algebra $(A(L),F)$ (briefly $A(L)$) is said to be a *fuzzy power of* $\mathcal{A}$.

2. In the following, each set $P = \{p_i;\ i \in I\}$ L of pairwise-disjoint elements, the supremum of which is 1, will be called *a partition in* L ($p_i \wedge p_j = 0$, $\vee\, p_i = 1$). (A partition is obviously a maximal orthogonal system in L.)

It is clear that for an $\overline{X} \in A(L)$, the set $\{X(a);\ a \in A\}$ (the image of $\overline{X}$) is a partition in L.

Theorem 1. *A complete lattice L allows the fuzzy power if and only if it satisfies the following three conditions:*

a) L is complemented;

b) If $\{p_i;\ i \in I\}$ *is a partition in L, then* $\{q_j;\ j \in J\}$ *is also a partition in L, where J is any set-theoretic partition of I, and for every* $j \in J$, $q_j = \bigvee_{k \in j} p_k$;

c) If $P_1,\dots,P_n$ *are partitions in L, then* $P = \{p_1 \wedge \dots \wedge p_n;\ p_i \in P_i,\ i = 1,\dots,n\}$ *is also a partition in L.*

Proof.

($\rightarrow$) Let L be a lattice which allows the fuzzy power. Then:

a) L is complemented. Indeed, by (*i*) every $p \in L$ belongs to the image of some $\overline{X}$, that is, to one partition, say $\{p_i;\ i \in I\}$. By (*ii*), for $n=1$, i.e. for unary operations, $\{p, \bigvee_{p_i \neq p} p_i\}$ is also a partition in L, and $\bigvee_{p_i \neq p} p_i$ is a complement of p.

b) Consider an arbitrary partition P. Since (*ii*) holds for any unary operation applied on some $\overline{X}$ such that $\{\overline{X}(a);\ a \in A\} = P$, the proof of this part follows immediately.

c) To prove the third property, consider an n-ary operation f, and fuzzy sets $\overline{X}_1,\dots,\overline{X}_n$, such that $\{\overline{X}_i(a);\ a\in A\} = P_i$, $i=1,\dots,n$. The proof now follows from the fact that the image of $f(\overline{X}_1,\dots,\overline{X}_n)$ has to be a partition.

($\leftarrow$) Let L now satisfy *a*), *b*) and *c*). Then, since L is complemented, (*i*) is satisfied in $A(L)$ for any algebra $\mathcal{A}$. If $f\in F_n$, and $\overline{X}_1,\dots,\overline{X}_n\in A(L)$, then (*ii*) holds, since by *b*) and by *c*) $f(\overline{X}_1,\dots,\overline{X}_n)$ is a partition in L. □

Considering lattices with finite partitions only, one can characterize those which allow the fuzzy power by means of 0,1-lattice homomorpshisms, i.e. using the congruences on L with one element minimal and maximal class ($[0]_\theta=\{0\}$, $[1]_\theta=\{1\}$), called in the following 0,1-congruences.

Theorem 2. *If a complete lattice L contains no infinite partition, then L allows the fuzzy power if and only if there is a 0,1-congruence θ on a semilattice* $(L,\wedge)$ *such that* $(L/\theta,\ \wedge\ ,\ \vee,{}',0\ ,\ 1\ \}$ *is a Boolean algebra, where*

$$(\bullet)\quad\left\{\begin{array}{l} [x]_\theta\vee[y]_\theta \overset{\text{def}}{=} \left\{\begin{array}{ll}[x]_\theta, & \text{if}\quad [x]_\theta=[y]_\theta\\ {[x\vee y]_\theta} & \textit{otherwise}\end{array}\right. ;\\[2ex] [x]'_\theta = [x']_\theta\ ; \end{array}\right.$$

"$\wedge$ " *is induced by* "$\wedge$" *in L.*

Proof.

($\leftarrow$) Suppose that there is a congruence θ on $(L,\wedge)$, satisfying ($\bullet$). Then, $\{p_1,\dots,p_n\}$ is a partition in L if and only if $\{[p_1]_\theta,\dots,[p_n]_\theta\}$ is a partition on L/θ. Indeed, $p\wedge q=0$ implies $[p]_\theta\neq[q]_\theta$, and thus

$$\bigvee_{i=1}^{n}[p_i]_\theta = \left[\bigvee_{i=1}^{n}p_i\right]_\theta = 1,\quad\text{and}\quad [p_i]_\theta\wedge\ [p_j]_\theta=[p_i\wedge q_j]_\theta=0\ .$$

On the other hand, if $\{[p_i]_\theta,\dots,[p_n]_\theta\}$ is a partition in L/θ, then $[p_i]_\theta\wedge\ [p_j]_\theta=0$ implies $[p_i\wedge p_j]_\theta=[0]_\theta$, i.e. $p_i\wedge p_j=0$, and $1=\bigvee_{i=1}^{n}[p_i]_\theta = \left[\bigvee_{i=1}^{n}p_i\right]_\theta=[1]_\theta=\{1\}$, that is $\bigvee_{i=1}^{n}p_i=1$, proving that $\{p_i,\dots,p_n\}$ is a partition in L.

Hence, it follows that L allow the power, since the properties *a*), *b*) and *c*) concerning the partitions, are satisfied in the Boolean algebra L/Θ, and thus in L.

($\rightarrow$) Suppose now that L allows the fuzzy power. Define a binary relation Θ on L: $(p,q) \in \Theta$ if and only if there is an $r \in L$, such that $\{p,r\}$ and $\{q,r\}$ are partitions in L. It is clear that Θ is an equivalence relation on L. It is also a congruence on $(L,\wedge)$:
For $i=1,2$, $(p_i,q_i) \in \Theta$ if and only if there is an $r_i \in L$, such that $\{p_i,r_i\}$ and $\{q_i,r_i\}$ are partitions in L. Then, by *c*) in Theorem 1, $\{p_1\wedge p_2,\ p_1\wedge r_2,\ p_2\wedge r_1,\ r_1,r_2\}$ and $\{q_1\wedge q_2,\ q_1\wedge r_2,\ q_2\wedge r_1,\ r_1\wedge r_2\}$ are both partitions in L. Again by *c*)
$\{p_1\wedge p_2\wedge q_1\wedge q_2,\ p_1\wedge q_1\wedge r_2,\ p_2\wedge q_2\wedge r_1,\ r_1\wedge r_2\}$ is a partition in L.
Moreover, $\{p_1\wedge p_2,\ (p_1\wedge q_1\wedge r_2)\vee(p_2\wedge q_2\wedge r_1)\vee(r_1\wedge r_2)\}$ and $\{q_1\wedge q_2,\ (p_1\wedge q_1\wedge r_2)\vee(p_2\wedge q_2\wedge r_1)\vee(r_1\wedge r_2)\}$ are also two partitions, by *b*) and by the fact that $p_1\wedge p_2 \ge p_1\wedge p_2\wedge q_1\wedge q_2$, $q_1\wedge q_2 \ge p_1\wedge p_2\wedge q_1\wedge q_2$. Thus, $(p_1\wedge p_2,\ q_1\wedge q_2) \in \Theta$.

Since $[0]_\Theta = \{p; (0,p) \in \Theta\}$, there is $q \in L$, such that $\{p,q\}$ is a partition, implying that $\{0,q\}$ is also a parition. Hence, $q=1$, $p=0$, i.e. $[0]_\Theta = \{0\}$.

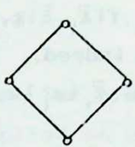
Similarly $[1]_\Theta = \{1\}$.

L/Θ is a a Boolean algebra. Indeed, L/Θ contains the least element 0, and the greatest one ▮. It is a uniquely complemented lattice, since for every $p \in L$, there is a complement $p' \in L$ (property *a*)), and by the definition of Θ, $[p']_\Theta$ contains all the complements of p. To prove that L/Θ is an atomic lattice, take one partition $P = \{p_1,\dots,p_n\}$ in L with a maximal number n of elements (by assumption n is finite). Then, $[P]_\Theta = \{[p_1]_\Theta,\dots,[p_n]_\Theta\}$ is the unique partition in L/Θ with a maximal (n again) number of elements. Indeed, if $Q = \{q_1,\dots,q_n\}$ is another partition in L, then $|P\cdot Q| = |\{p_i\wedge q_j;\ p_i \in P,\ q_j \in Q\}| = n$ (since otherwise $p_{i_1} \ge p_{i_1}\wedge q_{i_1},\dots,p_{i_k} \ge q_{i_k}$ $(k < n)$, and from $\bigvee_{t=1}^{k}(p_{i_t}\wedge q_{i_t}) = 1$, it follows that $\bigvee_{t=1}^{k} p_{i_t} = 1$, and hence $\{p_{i_1}\vee\dots\vee p_{i_k},\ P\setminus\{p_{i_1},\dots,p_{i_k}\}\}$ is not a partition in L, contrary to *b*)). Also, if $|Q| = n$, then $[P]_\Theta = [Q]_\Theta$, since $p_1 \ge p_1\wedge q_1$, $q_1 \ge p_1\wedge q_1,\dots,p_n \ge p_n\wedge q_n$, $q_n \ge p_r\wedge q_n$ and $[p_i]_\Theta = [q_i]_\Theta = [p_i\wedge q_i]_\Theta$, $i=1,\dots,n$.

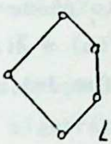
Now, if r is an arbitrary element from L, then $\{r, r'\} = R$ is a partition in L, and $[R\cdot P]_\Theta = [P]_\Theta$. Hence, either $r \wedge p_1 = 0$, or $[r \wedge p_1]_\Theta = [p_1]_\Theta$, i.e. $[r]_\Theta \wedge [p_1]_\Theta = [p_1]_\Theta$, proving that $[P]_\Theta$ is a set of atoms, and that L/Θ is atomic.

Since, L/Θ is atomic (with n atoms), and uniquely complemented, then it is distributive, and thus it is a (finite) Boolean algebra. □

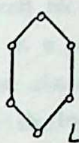
The Hasse-diagrams of some finite lattices allowing the fuzzy power are given below.



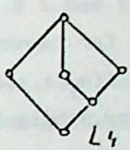
L_1



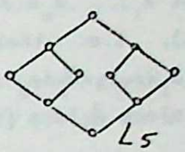
L_2



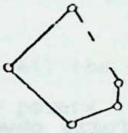
L_3



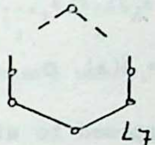
L_4



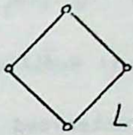
L_5



L_6

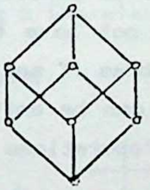


L_7

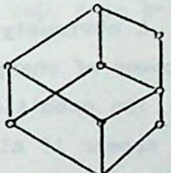


Li/Θ $i = 1, \ldots 7$

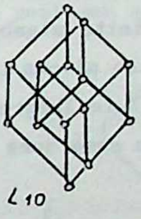
L_8.



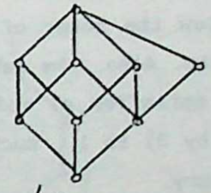
Li/Θ $i = 8, \ldots, 11$



L_9



L_{10}



L_{11}

3. The fuzzy power of an algebra $\mathcal{A}$ is defined on the collection of all the fuzzy sets on A. As it is known, we can consider some of these fuzzy sets as fuzzy subalgebras of $\mathcal{A}$ ([2]):

If $\mathcal{A}=(A, F)$ is an algebra and L a complete lattice, then a fuzzy set $\bar{A}\colon A \to L$ is said to be a *fuzzy sublagebra of* $\mathcal{A}$ if for every operation $f \in F_n \subseteq F$ and for $x_1, \ldots, x_n \in A$

$$\bar{A}(f(x_1, \ldots, x_n)) \geq \bigwedge_{i=1}^{n} \bar{A}(x_i) .$$

Is it possible to restrict a fuzzy power of an aglgebra to any of its fuzzy subalgebras? The answer is positive, as shown by the following theorem.

Theorem 3. *Let $\mathcal{A}$ be an algebra, and L a lattice allowing the fuzzy power. Let $\bar{A}$: $A \to L$ be a fuzzy subalgebra of $\mathcal{A}$, and*

$$\bar{A}(L) \overset{\text{def}}{=} \left\{ \bar{X} \in A(L);\ \bar{X}(a) \le \bar{A}(a), \text{ for every } a \in A \right\}$$

Then, $(\bar{A}(L), F)$ (briefly $\bar{A}(L)$) *is a subalgebra of the fuzzy power $A(L)$.*

(We shall call $\bar{A}(L)$ a *fuzzy power of a fuzzy (sub)algebra* $\bar{A}$.)

Proof.

Let $\bar{X}_1, \dots, \bar{X}_n \in \bar{A}(L)$ and $f \in F_n$. We have to prove that $f(\bar{X}_1, \dots, \bar{X}_n) = \bar{Y} \in \bar{A}(L)$, i.e. that for every $a \in A$, $\bar{Y}(a) \le \bar{A}(a)$. Indeed, $\bar{Y}(a) = \vee(\bar{X}_1(a_1) \wedge \dots \wedge \bar{X}_n(a_n));\ f(a_1, \dots, a_n) = a)$, and for $i = 1, \dots, n$, $\bar{X}_i(a_i) \le \bar{A}(a_i)$. Hence, since $\bar{A}$ is a fuzzy subalgebra of $\mathcal{A}$,

$$\bar{Y}(a) = \vee\,(\bar{A}(a_1) \wedge \dots \wedge \bar{A}(a_n);\ f(a_1, \dots a_n) = a) \le$$

$$\le \bar{A}(f(a_1, \dots, a_n)) = \bar{A}(a). \ \square$$

4. Lattices characterized in 2 are supposed to allow the fuzzy power of any algebra. If we restrict our attention to some special classes of algebras, then we can omit some of the required conditions. In particular, to allow the power of finite algebras, L obviously does not have to be complete. Also, the lattice allows the power of unary algebras if and only if it satisfies *a)* and *b)* in Theorem 1. The condition *c)* can be omitted, since by 3) in 1., such a case does not appear if all the operations on $\mathcal{A}$ are unary.

In the following, we shall give some general properties of fuzzy powers.

Theorem 4. *a) Let $f, g \in F_n$, and $h \in F_1$. Then, the following identities are preserved under the construction of fuzzy powers:*

a) $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$;

b) $g(x_1, \dots, x_n) = h(f(x_1, \dots, x_n))$.

Proof.

a) Let $a \in A$, and $\bar{X}_1, \dots, \bar{X}_n \in A(L)$. Then,

$f(\bar{X}_1, \dots, \bar{X}_n)(a) = \vee(\bar{X}_1(a_1) \wedge \dots \wedge \bar{X}_n(a_n)\ ;\ f(a_1, \dots, a_n = a) =$

$\vee(\bar{X}_1(a_1) \wedge \dots \wedge \bar{X}_n(a_n)\ ;\ g(a_1, \dots, a_n = a) = g(\bar{X}_1, \dots, \bar{X}_n)(a).$

Thus, $f(\overline{X}_1,\dots,\overline{X}_n) = g(\overline{X}_1,\dots,\overline{X}_n)$.

b) Again, let $a \in A$, $\overline{X}_1,\dots,\overline{X}_n \in A(L)$. Then,

$$h(f(\overline{X}_1,\dots,\overline{X}_n))(a) = \vee(f(\overline{X}_1,\dots,\overline{X}_n)(b) \;;\; h(b) = a) =$$

$$\vee(\vee(\overline{X}_1(b_1)\wedge\dots\wedge\overline{X}_n(b_n) \;;\; f(b_1,\dots,b_n = b) \;;\; h(b) = a) =$$

$$\vee(\overline{X}_1(b_1)\wedge\dots\wedge\overline{X}_n(b_n) \;;\; h(f(b_1,\dots,b_n)) = a) =$$

$$\vee(\overline{X}_1(b_1)\wedge\dots\wedge\overline{X}_n(b_n) \;;\; g(b_1,\dots,b_n) = a) = g(\overline{X}_1,\dots,\overline{X}_n)(a).$$

Hence $g(\overline{X}_1,\dots,\overline{X}_n) = h(f(\overline{X}_1,\dots,\overline{X}_n))$. □

Remark. The previous proposition shows that the fuzzy power gives the most if applied on unary algebras, since by b) any identity of the form

$$f_{i_1}\dots f_{i_m}(x) = f_{j_1}\dots f_{j_n}(x),$$

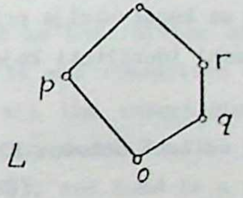
where all the operations are unary, is preserved under the construction of fuzzy powers.

5. We shall conclude with a simple example of a fuzzy power, the algebra being a two-element group. L is here a pentagon η_5.

Example.

A:

·	e	a
e	e	a
a	a	e



$$A(L) = \left\{\left\{\begin{bmatrix} e & a \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} e & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e & a \\ p & q \end{bmatrix}, \begin{bmatrix} e & a \\ q & p \end{bmatrix}, \begin{bmatrix} e & a \\ p & r \end{bmatrix}, \begin{bmatrix} e & a \\ r & p \end{bmatrix}\right\}, \cdot\right\}$$

Denote $\begin{bmatrix} e & a \\ p & q \end{bmatrix}$ by pq, etc.

·	10	01	pq	qp	pr	rp
10	10	01	pq	qp	pr	rp
01	01	10	qp	pq	rp	pr
pq	pq	qp	10	01	10	01
qp	qp	pq	01	10	01	10
pr	pr	rp	10	01	10	01
rp	rp	pr	01	10	01	10

(Operation on $A(L)$).

One fuzzy subalgebra $\bar{A}$ of A is $\bar{A} = \begin{bmatrix} e & a \\ 1 & r \end{bmatrix}$. The corresponding fuzzy power is the following collection of mappings:

$$\bar{A}(L) = \left\{ \left\{ \begin{bmatrix} e & a \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} e & a \\ p & q \end{bmatrix}, \begin{bmatrix} e & a \\ p & r \end{bmatrix} \right\}, \ \cdot \ \right\}$$

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Rezime

RASPLINUTI STEPEN ALGEBRI

Pokazano je da se jedna klasa komplementiranih mreža može iskoristiti za konstrukciju mrežnih stepena algebri, kao uopštenja Bulovih stepena. Pokazano je da se konstrukcija prirodno proširuje na rasplinute podalgebre i ispitani su neki identiteti koje se pri konstrukciji očuvavaju.

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CEP AND HOMOMORPHIC IMAGES OF ALGEBRAS

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Abstract

The CEP (congruence extension property) of an algebra is characterized by means of modular pairs in its lattice of weak congruences. Hence the necessary and sufficent conditions are deduced under which all homomorphic images of an algebra having the CEP, satisfy the same property. This characterization is purely lattice theoretic, so two special cases are considered for which some sufficent conditions (modularity of congruence lattice, for example) are given.

Introduction

If $\mathcal{A} = (A,F)$ is an algebra, then $C_w\mathcal{A}$ is the lattice of all the weak congruences on $\mathcal{A}$, i.e. of all the symmetric and transitive subalgebras of $\mathcal{A}^2$. $C_w\mathcal{A}$ coincides with the lattice of all the congruences on all the subalgebras of $\mathcal{A}$ (under set inclusion). Moreover, Con$\mathcal{A}$ is its sublattice (as a filter generated by $\Delta = \{(x,x) \mid x \in B\}$), and Sub$\mathcal{A}$ is a retract of $C_w\mathcal{A}$ (as an ideal generated by Δ). In particular, subalgebras are represented by diagonal relations in $C_w\mathcal{A}$ ($B \in$ Sub$\mathcal{A}$ corresponds to $d_{B^2} = \{(x,x) \mid x \in B\}$). (For the notation, definitions, and other properties of $C_w\mathcal{A}$, see [1], [2], [3], and the references given there).

If L is any lattice, and $a,b \in L$, then (a,b) is said to be the *modular pair* in L (see for example [4]), if for every $x \in L$,

$$x \le b \text{ implies } x \vee (a \wedge b) = (x \vee a) \wedge b.$$

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Key words: CEP, weak congruences.

A modular pair (a,b) is often denoted by aMb. Obviously, L is modular if and only if aMb for all $a,b \in L$.

Theorem 1. *An algebra $\mathcal{A}$ has the CEP if and only if for every $B \in \mathrm{Sub}\mathcal{A}$, $\Delta M B^2$ in $C_w\mathcal{A}$* (that is, iff for every $\rho \in C_w\mathcal{A}$

$$\rho \le B^2 \text{ implies } \rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2,$$

where $\Delta = \{(x,x) \mid x \in \mathcal{A}\}$).

Proof.

It is well known that $\mathcal{A}$ has the CEP if every congruence ρ on a subalgebra B of $\mathcal{A}$ can be extended to the congruence θ on $\mathcal{A}$, i.e. iff

$$\rho = \theta \cap B^2 \text{ , which is equivalent to}$$

$$(1) \qquad \rho = (\rho \vee \Delta) \wedge B^2 \quad \text{in } C_w\mathcal{A}.$$

Indeed, if there is such a θ, then there is at least one with the same property, namely $\theta = \rho \vee \Delta$; on the other hand, if (1) is satisfied, then $\theta = \rho \vee \Delta$.

Now, (1) is equivalent to

$$(2) \qquad \rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2,$$

since $\rho \ge \Delta \wedge B^2$, and hence $\rho = \rho \vee (\Delta \wedge B^2)$.

Thus, for $\rho \in \mathrm{Con}B$, the CEP is equivalent to

$$(3) \qquad \rho \le B^2 \rightarrow \rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2 .$$

Even in the case when $\rho \in \mathrm{Con}C$, $C < B$ (we still have $\rho \le B^2$), the CEP implies (3). Indeed, if $\sigma = \rho \vee (\Delta \wedge B^2)$ (which belongs to $\mathrm{Con}B$), then by the CEP

$$\sigma = (\sigma \vee \Delta) \wedge B^2, \text{ i.e.}$$

$$\rho \vee (\Delta \wedge B^2) = (\rho \vee (\Delta \wedge B^2) \vee \Delta) \wedge B^2, \text{ and}$$

since $\rho \vee \Delta = (\rho \vee (\Delta \wedge B^2)) \vee \Delta$ (see [1]), finally we have

$$\rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2 .$$

Thus, the CEP implies (3) for every $\rho \in C_w\mathcal{A}$. Since the converse has already been proved, we are done. □

It is well known that for $\theta \in \mathrm{Con}\mathcal{A}$, the lattice $\mathrm{Con}\mathcal{A}/\theta$ is isomorphis with the filter $[\theta)$ in $\mathrm{Con}\mathcal{A}$, under $\rho \to \rho/\theta$, where $[a]_\theta(\rho/\theta)[b]_\theta$ iff $(a,b) \in \rho$. Hence, the lattice $C_w\mathcal{A}/\theta$ ($\theta \in \mathrm{Con}\mathcal{A}$) consists (up to the above isomorphism) of all the intervals $[\theta \wedge B^2, B^2]$ in $C_w\mathcal{A}$, with the property $B[\theta]$

isomorphism) of all the intervals $[\Theta \wedge B^2, B^2]$ in $C_w\mathcal{A}$, with the property $B[\Theta] = B$, where $B[\Theta] = \{x \in A \mid x\Theta b \text{ for some } b \in B\}$.

In the following we shall identify those intervals and their inverse images.

Theorem 2. *Let $\Theta \in \mathrm{Con}\mathcal{A}$. Then $\mathcal{A}/\Theta$ has the CEP if and only if in the lattice $C_w\mathcal{A}\Theta M B^2$, for every $B \in \mathrm{Sub}\mathcal{A}$ satisfying $B = B[\Theta]$.* (In other word, $\mathcal{A}/\Theta$ has the CEP iff for every such $\mathcal{B}, \rho \le B^2$ implies $\rho \vee (\Theta \wedge B^2) = = (\rho \vee \Theta) \wedge B^2$).

Proof. By the above mentioned isomorphism, $\mathcal{A}/\Theta$ has the CEP if and only if for every $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, such that $B = B[\Theta]$, and for every $\rho \in [B^2 \wedge \Theta, B^2]$, there is $\sigma \in [\Theta)$ such that $\rho = \sigma \cap B^2$, i.e. if and only if

(1) $$\rho = (\rho \vee \Theta) \wedge B^2 \text{ in } C_w\mathcal{A}.$$

Obviously, $\rho = \rho \vee (\Theta \wedge B^2)$, and thus (1) is equivalent with

(2) $$\rho \vee (\Theta \wedge B^2) = (\rho \vee \Theta) \wedge B^2,$$

where

$$B^2 \wedge \Theta \le \rho \le B^2 .$$

If ρ is any (weak) congruence satisfying $\rho \le B^2$, then, obviously

(3) $$\rho \vee \Theta = (\rho \vee (\Theta \wedge B^2)) \vee \Theta .$$

Now, if $\mathcal{A}/\Theta$ has the CEP, then for $\sigma = \rho \vee (\Theta \wedge B^2)$, we have

$$\sigma = (\sigma \vee \Theta) \wedge B^2, \quad \text{i.e.}$$

$$\rho \vee (\Theta \wedge B^2) = (\rho \vee (\Theta \wedge B^2) \vee \Theta) \wedge B^2 .$$

Hnece, by (3)

$$\rho \vee (\Theta \wedge B^2) = (\rho \vee \Theta) \wedge B^2 , \text{ for } \rho \le B^2 ,$$

which was to be proved. □

Since every homomorphic image of $\mathcal{A}$ is isomorphic with $\mathcal{A}/\Theta$ for some $\Theta \in \mathrm{Con}\mathcal{A}$, Theorem 2 gives a characterization of the CEP for any homomorphic image of $\mathcal{A}$. To extend this characterization on every interval $[\Theta \wedge B^2, B^2]$ in $C_w\mathcal{A}$ (not only on those satisfying condition $B = B[\Theta]$), we need the following lemma.

Lemma 3. *Let $\mathcal{A}$ have the CEP, and let $\mathcal{B},\mathcal{C} \in \mathrm{Sub}\mathcal{A}$, $\mathcal{B} < \mathcal{C}$. If $\rho \in \mathrm{Con}\mathcal{B}$, $\theta \in \mathrm{Con}\mathcal{A}$, and if $\rho < B^2 \wedge (\rho \vee \theta)$, then $\rho \vee (\Delta \wedge C^2) < C^2 \wedge (\rho \vee \theta)$, all in the lattice $C_w\mathcal{A}$.*

Proof. Suppose that

$$(1) \qquad \rho \vee (\Delta \wedge C^2) = C^2 \wedge (\rho \vee \theta) .$$

By the CEP, and since $\mathcal{B} < \mathcal{C}$, we have

$$(2) \qquad \rho = B^2 \wedge (\rho \vee (\Delta \wedge C^2)) .$$

From (2) and (1), we finally get

$$\rho = B^2 \wedge C^2 \wedge (\rho \vee \theta) = B^2 \wedge (\rho \vee \theta) .$$

The proof now follows by contraposition. □

Theorem 4. *Let $\mathcal{A}$ be an algebra satisfying $B[\theta] < A$ for every $\theta \in \mathrm{Con}\mathcal{A}$, $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, $\theta \neq A^2$, $B \neq A$. Then, every homomorphic image of $\mathcal{A}$ has the CEP, if and only if $\theta\, M\, B^2$ for every $\theta \in \mathrm{Con}\mathcal{A}$, and for every $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$ (all in $C_w\mathcal{A}$).*

Proof. By the proof of Theorem 2, the CEP on $\mathcal{A}/\theta$ is equivalent with $\theta M B^2$ for every $\mathcal{B}$ such that $B = B[\theta]$. Now, if $\theta M B^2$ is not satisfied for some subalgebra $\mathcal{B}$, such that $B < B[\theta]$, then it is not true that $\theta M C^2$ for any $C \in \mathrm{Sub}\mathcal{A}$, such that $\mathcal{B} \leq C$ (by assumption on $\mathcal{A}$, such a $C \neq \mathcal{A}$ always exists). This contradicts Theorem 2, proving that $\theta\, M\, B^2$ for every $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$. □

It is possible to connect the given lattice characterizations of the CEP on homomorphic images of $\mathcal{A}$, with some known properties of $\mathrm{Con}\mathcal{A}$. But first, we need the following definition.

An algebra $\mathcal{A}$ has the *weak congruence intersection property* (WCIP), if for $\rho \in C_w\mathcal{A}$ the following implication holds in $C_w\mathcal{A}$:

$$\Delta \leq \theta \longrightarrow \Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge \theta .$$

Obviously, $\Delta \leq \theta$ means that $\theta \in \mathrm{Con}\mathcal{A}$.

Remark. The WCIP is a weakened *congruence intersection property* (CIP); $\mathcal{A}$ has the CIP if for all $\rho,\theta \in C_w\mathcal{A}$, the following equality holds in $C_w\mathcal{A}$:

$$(\rho \wedge \theta) \vee \Delta = (\rho \vee \Delta) \wedge (\theta \vee \Delta) .$$

The CIP was defined in [1] as one of the necessary and sufficent conditions for the modularity of $C_w\mathcal{A}$.

It is clear that the CIP implies the WCIP, since for $\Theta \in \mathrm{Con}\mathcal{A}$, $\Theta \vee \Delta = \Theta$.

Theorem 5. *If $\mathcal{A}$ has the WCIP and $\mathrm{Con}\mathcal{A}$ is modular, then the CEP is hereditary for homomorphic images of $\mathcal{A}$.*

Proof. Let $\rho \le B^2$, $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, $\Theta \in \mathrm{Con}\mathcal{A}$. Then,

$(\rho \vee (\Theta \wedge B^2)) \vee \Delta = (\rho \vee \Delta) \vee ((\Theta \wedge B^2) \vee \Delta) =$ (by WCIP) $=$

$(\rho \vee \Delta) \vee (\Theta \wedge (B^2 \vee \Delta)) =$ (since $\mathrm{Con}\mathcal{A}$ is modular) $=$

$((\rho \vee \Delta) \vee \Theta) \wedge (B^2 \vee \Delta) = (\rho \vee (\Delta \vee \Theta)) \wedge (B^2 \vee \Delta) =$

$(\rho \vee \Theta) \wedge (B^2 \vee \Delta) =$ (again by WCIP) $= ((\rho \vee \Theta) \wedge B^2) \vee \Delta$.

$\mathcal{A}$ has the CEP, and $\rho \vee (\Theta \wedge B^2)$ as well as $(\rho \vee \Theta) \wedge B^2$ belong to $\mathrm{Con}\mathcal{B}$. $\rho \to \rho \vee \Delta$ is then an injection (see [1]), and thus

$$\rho \vee (\Theta \wedge B^2) = (\rho \vee \Theta) \wedge B^2 \ . \ \square$$

The CEP can be hereditary for homomorphic images even if $\mathrm{Con}\mathcal{A}$ is not modular. To prove this, we need the following lemma.

Lemma 6. *If $\mathcal{A}$ has the CEP and WCIP, then for $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, $\Theta \in \mathrm{Con}\mathcal{A}$, the following implication holds in $C_w\mathcal{A}$:*

$$\Theta \le B^2 \vee \Delta \longrightarrow \textit{for } \rho \in [B^2 \wedge \Theta, B^2],\ \rho \vee \Theta = \rho \vee \Delta \ .$$

Proof. If $\Theta \le B^2 \vee \Delta$ and $\rho \in [B^2 \wedge \Theta, B^2]$, then

$$\Theta = (\Delta \vee B^2) \ \ \Theta = \text{(by WCIP)} = \Delta \vee (B^2 \wedge \Theta) \le \rho \vee \Delta \ .$$

Hence $\rho \vee \Delta \vee \Theta = \rho \vee \Delta$, i.e. $\rho \vee \Theta = \rho \vee \Delta$. $\square$

Theorem 7. *Let $\mathcal{A}$ have the WCIP, and for every $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$ $\mathrm{Con}\mathcal{A} \setminus [B^2 \vee \Delta) = (B^2 \vee \Delta]$, then the CEP is hereditary for homomorphic images of $\mathcal{A}$.* ($[B^2 \vee \Delta)$ is a filter and $(B^2 \vee \Delta]$ an ideal generated by $B^2 \vee \Delta$ respectively, both in $\mathrm{Con}\mathcal{A}$.)

Proof. $\mathcal{A}$ has the WCIP, and since nontrivial factor-algebras are generated by congruences less than $B^2 \vee \Delta$ only, the conditions of Lemma 6 are satisfied. Hence, for $\Theta \in \mathrm{Con}\mathcal{A}$, $\Theta \le B^2 \vee \Delta$, $\rho \in \mathrm{Con}\mathcal{B}$, since $\Theta \wedge B^2 \le \rho$,

$$(\rho \vee \Theta) \wedge B^2 = (\rho \vee \Delta) \wedge B^2 = \rho = \rho \vee (\Theta \wedge B^2) \ . \ \square$$

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Rezime

CEP I HOMOMORFNE SLIKE ALGEBRI

U radu se daju potrebni i dovoljni uslovi da se svojsvo proširenja kongruencija (CEP) prenese sa algebre na sve njene homomorfne slike. Uslovi su formulisani mrežnim zakonima u mreži slabih kongruencija algebre. Daju se i neke posledice tih uslova.

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GENERALIZED PSEUDO-BOOLEAN FUNCTIONAL EQUATIONS OF THE THIRD ORDER

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Abstract

Necessary and sufficient conditions for existence of solutions to the third order pseudo-Boolean functional equation are given as well as the solutions in explicit form.

Introduction

Let $(P,+,\cdot)$ be a commutative ring with identity element 1 without divisors of zero and let L be a finite set. A generalized pseudo-Boolean function (GPB function) is every mapping f of L^n into P, i.e. $f: L^n \to P$, where L^n is a direct power of L.

The definiton of partial derivatives of GPB functions and some properties of these partial derivatives are given by Gilezan in [4].

Definition 1. *A partial derivative of a GPB function* $f: L^n \to P$ *with the variables* x_i $(i=1,2,\dots,n)$ *are GPB functions*

$$\frac{\partial f_a}{\partial x_i} : L^n \longrightarrow P \quad \textit{defined by}$$

$$(i) \quad \frac{\partial f_a}{\partial x_i}(X) = f(x_1,\dots,x_{i-1}, a, x_{i+1},\dots,x_n) - f(x), \quad a \in L$$

$$(1 \le i \le n), \qquad \textit{where} \quad X = (x_1, x_2, \dots, x_n).$$

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Key words and phrases: pseudo-Boolean functions, partial derivatives of pseudo Boolean functions, functional equations of pseudo-Boolean functions

A partial derivative of higher order of a GPB function is

$$(1')\qquad \frac{\partial^m f_{a_{i_1}\dots a_{i_m}}}{\partial x_{i_1}\dots\partial x_{i_m}} = \frac{\partial}{\partial x_{i_m}}\dots\left(\frac{\partial}{\partial x_{i_2}}\left(\frac{\partial f_{a_{i_1}}}{\partial x_{i_1}}\right)a_{i_2}\right)\dots)a_{i_m}$$

$$a_{i_j} \in L, \quad j=1,2,\dots,m \qquad (m \geq 1)$$

Directly from definition *1* follows that for every $\alpha, \beta, \in L$, for every $c \in P$ and for every couple of GPB functions f and g the following properties hold

$$\frac{\partial\, c\alpha}{\partial\, x_i}\; 0\,; \quad \frac{\partial\,(cf)\alpha}{\partial\, x_i} = c\,\frac{\partial\, f\alpha}{\partial\, x_i}\,; \quad \frac{\partial(f+g)}{\partial\, x_i} = \frac{\partial\, f\alpha}{\partial\, x_i} + \frac{\partial\, g\alpha}{\partial\, x_i}\,;$$

$$\frac{\partial(f\cdot g)}{\partial\, x_i} = \frac{\partial\, f\alpha}{\partial\, x_i}\cdot g + f\,\frac{\partial\, g\alpha}{\partial\, x_i} + \frac{\partial\, f\alpha}{\partial\, x_i}\cdot\frac{\partial\, g\alpha}{\partial\, x_i}\,;$$

$$\frac{\partial^2 f\alpha\beta}{\partial\, x_i\cdot\partial\, x_j} = \frac{\partial^2 f\beta\alpha}{\partial\, x_j\cdot\partial\, x_i}\,, \qquad i\neq j\,;$$

$$\frac{\partial^m f\alpha^m}{\partial\, x_i^m} = (-1)^{m+1}\,\frac{\partial\, f\alpha}{\partial\, x_i}\,, \quad 1 \leq m \leq n \quad m \text{ is a natural number, } 1 \leq i \leq n.$$

However, the relation

$$F\left(g_1, g_2, \dots, g_k,\ f,\ \frac{\partial\, f\alpha_1}{\partial\, x_1}, \dots, \frac{\partial\, f\alpha_n}{\partial\, x_n}, \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2}, \dots\right.$$

$$\left.\frac{\partial^2 f\alpha_{n-1}\alpha_n}{\partial x_{n-1}\partial x_n}, \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3}, \dots, \frac{\partial^3 f\alpha_{n-2}\alpha_{n-1}\alpha_n}{\partial x_{n-2}\partial x_{n-1}\partial x_n}, \dots, \frac{\partial^n f\alpha_1\dots\alpha}{\partial x_1\dots\,\partial x_n}\right) = 0$$

$\alpha_1, \dots, \alpha_n \in L$ is a generalized pseudo-Boolean functional equation, where $g_1, g_2, \dots, g_n$ known functions, an unknown function f and some of its partial derivatives take place. Hence, the solution of this functional equation has to be found.

Lemma 1. *A functional equation with an unknown GPB function f*

$$\frac{\partial\, f\alpha_i}{\partial\, x_i} = g(x), \quad \text{where} \quad \alpha_i \in L,$$

has a solution if and only if

$$g(x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n) = 0.$$

The solutions f are determined by the following formula

$$f(X) = c(x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n) - g(X) \quad \text{or else}$$

$$\int_{\alpha_i} g(X)\, dx_i = c(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) - g(X)$$

where c is an arbitrary function of the variables $x_1,\dots,x_n$.

The proof of this lemma is given in [5].

Let us introduce the following notation

$$(\tilde{x}_i) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \quad \text{and}$$

$$(\tilde{\tilde{\alpha}}_i) = (x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n).$$

Theorem 1. *A system of GPB functional equations*

1.1. $\dfrac{\partial f\alpha_i}{\partial x_i} = P_i(X) \qquad i=1,2,\dots,n, \qquad X = (x_1,\dots,x_n)$

has a solution if and only if

1.2. $P_i(\tilde{\tilde{\alpha}}_i) = 0 \quad i=1,2,\dots,n$

1.3. $\dfrac{\partial P_j\alpha_i}{\partial x_i} = \dfrac{\partial P_i\alpha_j}{\partial x_j}, \qquad i,j = 1,2,\dots,n \quad i \neq j.$

The solutions f are determined by the following

Formula (they are equivalent to each other)

2.1. $f(X) = c - \sum_{k=1} P_{i_k}(\tilde{\tilde{\alpha}}_{i_{k-1}}, \tilde{\tilde{\alpha}}_{i_{k-1}}, \dots, \tilde{\tilde{\alpha}}_{i_n}) - P_{i_n}(X)$

where $i_1 i_2 \dots i_n$ are permutations of the set $\{1,2,\dots n\}$ and c is an arbitrary constant from P. The proof of this theorem is given in [5].

Lemma 2. *A system of GPB functional equations*

$$\frac{\partial f\alpha_1}{\partial x_1} = X(x_1,x_2,x_3), \quad \frac{\partial f\alpha_2}{\partial x_2} = Y(x_1,x_2,x_3), \quad \frac{\partial f\alpha_3}{\partial x_3} = Z(x_1,x_2,x_3),$$

has a solution if and only if

$$X(\alpha_1, x_1, x_3) = Y(x_1, \alpha_2, x_3) = Z(x_1, x_2, \alpha_3) = 0 \tag{1}$$

$$\frac{\partial X\alpha_2}{\partial x_2} = \frac{\partial Y\alpha_1}{\partial x_1} \text{ and } \frac{\partial X\alpha_3}{\partial x_3} = \frac{\partial Z\alpha_1}{\partial x_1} \text{ and } \frac{\partial Y\alpha_3}{\partial x_3} = \frac{\partial Z\alpha_2}{\partial x_2} \tag{2}$$

All the function solutions are determined by the following formula

$$\begin{aligned} f(x_1, x_2, x_3) = C - X(x_1, x_2, x_3) - Y(x_1, x_2, x_3) - Z(x_1, x_2, x_3) - \\ \frac{\partial Y\alpha_3}{\partial x_3} - \frac{\partial X\alpha_2}{\partial x_2} - \frac{\partial X\alpha_3}{\partial x_3} - \frac{\partial^2 X\alpha_2\alpha_3}{\partial x_2 \partial x_3} \end{aligned} \tag{3}$$

The proof of this lemma follows from Theorem 1.

Let us consider a functional equation of the following form

$$\begin{aligned} F:\ a\frac{\partial f\alpha_1}{\partial x_1} + b\frac{\partial f\alpha_2}{\partial x_2} + c\frac{\partial f\alpha_3}{\partial x_3} + d\frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} + e\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \\ + g\frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} + h\frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = s(x_1, x_2, x_3) \end{aligned} \tag{4}$$

a, b, c, d, e, g, h and s are GPB functions which mapp $L^3 \to P$.

Theorem 2. *A functional equation (4) has a solution if and only if*

(5)

$$(u)\quad s + \frac{\partial s\alpha_1}{\partial x_1} + \frac{\partial s\alpha_2}{\partial x_2} + \frac{\partial s\alpha_3}{\partial x_3} + \frac{\partial^2 s\alpha_1\alpha_2}{\partial x_1 \partial x_2} + \frac{\partial^2 s\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \frac{\partial^3 s\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = 0$$

$$\begin{aligned} (u')\quad (a+b+c-g-e-d+h) \left[a + \frac{\partial a\alpha_2}{\partial x_2} + \frac{\partial a\alpha_3}{\partial x_3} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3}\right]\left[b + \frac{\partial b\alpha_1}{\partial x_1} + \frac{\partial b\alpha_3}{\partial x_3} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3}\right] \cdot \\ \left[c + \frac{\partial c\alpha_1}{\partial x_1} + \frac{\partial c\alpha_2}{\partial x_2} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2}\right]\left[d + \frac{\partial d\alpha_3}{\partial x_3} - a - \frac{\partial a\alpha_3}{\partial x_3} - b - \frac{\partial b\alpha_3}{\partial x_3}\right]\left[e + \frac{\partial e\alpha_2}{\partial x_2} - a - \right. \\ \left. - \frac{\partial a\alpha_2}{\partial x_2} - c - \frac{\partial c\alpha_2}{\partial x_2}\right]\left[g + \frac{\partial g\alpha_1}{\partial x_1} - b - \frac{\partial b\alpha_1}{\partial x_1} - c - \frac{\partial c\alpha_1}{\partial x_1}\right] = 1. \end{aligned}$$

If the conditions (5) (u) and (5)(u') are fulfilled, then all the functions *f* are determined by the formula (3), where, *X*, *Y* and *Z* are given further on the pages 9, 10, 11, respectively.

Proof. **If the partial derivatives of the equation (3) are found, i.e.**

$$, \frac{\partial F\alpha_1}{\partial x_1}, \frac{\partial F\alpha_2}{\partial x_2}, \frac{\partial F\alpha_3}{\partial x_3}, \frac{\partial^2 F\alpha_1\alpha_2}{\partial x_1 \partial x_2}, \frac{\partial^2 F\alpha_1\alpha_3}{\partial x_1 \partial x_3}, \frac{\partial^2 F\alpha_2\alpha_3}{\partial x_2 \partial x_3} \text{ and } \frac{\partial^3 F\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3}$$

then the following system of equations is obtained

$$\frac{\partial F\alpha_1}{\partial x_1}, \frac{\partial F\alpha_2}{\partial x_2}, \frac{\partial F\alpha_3}{\partial x_3}, \frac{\partial^2 F\alpha_1\alpha_2}{\partial x_1 \partial x_2}, \frac{\partial^2 F\alpha_1\alpha_3}{\partial x_1 \partial x_3}, \frac{\partial^2 F\alpha_2\alpha_3}{\partial x_2 \partial x_3} \text{ and } \frac{\partial^3 F\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3}$$

then the following system of equations is obtained

$$a \frac{\partial f\alpha_1}{\partial x_1} + b \frac{\partial f\alpha_2}{\partial x_2} + c \frac{\partial f\alpha_3}{\partial x_3} + d \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} + e \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + g \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} +$$

$$+ g \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = s$$

$$(6) \quad - a \frac{\partial f\alpha_1}{\partial x_1} + \frac{\partial b\alpha_1}{\partial x_1} \frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial c\alpha_1}{\partial x_1} \frac{\partial f\alpha_3}{\partial x_3} + \left[b + \frac{\partial b\alpha_1}{\partial x_1} - d\right] \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} +$$

$$+ \left[c + \frac{\partial c\alpha_1}{\partial x_1} - e\right] \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \frac{\partial g\alpha_1}{\partial x_1} \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} + \left[g + \frac{\partial g\alpha_1}{\partial x_1} - h\right] \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial s\alpha_1}{\partial x_1}$$

$$\frac{\partial a\alpha_2}{\partial x_2} \frac{\partial f\alpha_1}{\partial x_1} - b \frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial c\alpha_2}{\partial x_2} \frac{\partial f\alpha_3}{\partial x_3} + \left[a + \frac{\partial a\alpha_2}{\partial x_2} - d\right] \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} + \frac{\partial e\alpha_2}{\partial x_2} \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} +$$

$$+ \left[c + \frac{\partial c\alpha_2}{\partial y} - g\right] \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} + \left[e + \frac{\partial e\alpha_2}{\partial x_2} - h\right] \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial s\alpha_2}{\partial x_2},$$

$$\frac{\partial a\alpha_3}{\partial x_3} \frac{\partial f\alpha_1}{\partial x_1} + \frac{\partial b\alpha_3}{\partial x_3} \frac{\partial f\alpha_2}{\partial x_2} - c \frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial d\alpha_3}{\partial x_3} \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} + \left[a + \frac{\partial a\alpha_3}{\partial x_3} - e\right] \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} +$$

$$+ \left[b + \frac{\partial b\alpha_3}{\partial x_3} - g\right] \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} + \left[d + \frac{\partial d\alpha_3}{\partial x_3} - h\right] \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial s\alpha_3}{\partial x_3},$$

$$- \frac{\partial a\alpha_2}{\partial x_2} \frac{\partial f\alpha_1}{\partial x_1} - \frac{\partial b\alpha_1}{\partial x_1} \frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2} \frac{\partial f\alpha_3}{\partial x_3} + \left[-a - \frac{\partial a\alpha_2}{\partial x_2} - b - \frac{\partial b\alpha_1}{\partial x_1} + d\right] \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} +$$

$$+ \left(\frac{\partial c\alpha_2}{\partial x_2} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2} - \frac{\partial e\alpha_2}{\partial x_2}\right) \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \left(\frac{\partial c\alpha_1}{\partial x_1} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2} - \frac{\partial g\alpha_1}{\partial x_1}\right) \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \qquad (6)$$

$$+ \left[c + \frac{\partial c\alpha_2}{\partial x_2} + \frac{\partial c\alpha_1}{\partial x_1} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2} - g - \frac{\partial g\alpha_1}{\partial x_1} + h\right] \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial^2 s\alpha_1\alpha_2}{\partial x_1 \partial x_2} ,$$

$$- \frac{\partial a\alpha_3}{\partial x_3} \frac{\partial f\alpha_1}{\partial x_1} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} \frac{\partial f\alpha_2}{\partial x_2} - \frac{\partial c\alpha_1}{\partial x_1} \frac{\partial f\alpha_3}{\partial x_3} + \left(\frac{\partial b\alpha_3}{\partial x_3} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} - \frac{\partial d\alpha_3}{\partial x_3}\right) \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} +$$

$$\left[-a - \frac{\partial a\alpha_3}{\partial x_3} - c - \frac{\partial c\alpha_1}{\partial x_1} + e\right] \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \left(\frac{\partial b\alpha_1}{\partial x_1} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} - \frac{\partial g\alpha_1}{\partial x_1}\right) \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} +$$

$$+ \left[b + \frac{\partial b\alpha_3}{\partial x_3} + \frac{\partial b\alpha_1}{\partial x_1} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} - d - \frac{\partial d\alpha_3}{\partial x_3} - g \frac{\partial g\alpha_1}{\partial x_1} + h\right] \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial^2 s\alpha_1\alpha_3}{\partial x_1 \partial x_3} ,$$

$$\frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3} \frac{\partial f\alpha_1}{\partial x_1} - \frac{\partial b\alpha_3}{\partial x_3} \frac{\partial f\alpha_2}{\partial x_2} - \frac{\partial c\alpha_2}{\partial x_2} \frac{\partial f\alpha_2}{\partial x_2} + \left(\frac{\partial a\alpha_3}{\partial x_3} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3} - \frac{\partial d\alpha_3}{\partial x_3}\right) \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} +$$

$$+ \left(\frac{\partial a\alpha_2}{\partial x_2} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3} - \frac{\partial e\alpha_2}{\partial x_2}\right) \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \left[-b - c - \frac{\partial b\alpha_3}{\partial x_3} \frac{\partial c\alpha_2}{\partial x_2} + g\right] \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} +$$

$$+ \left[a + \frac{\partial a\alpha_3}{\partial x_3} + \frac{\partial a\alpha_2}{\partial x_2} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3} - d - \frac{\partial d\alpha_3}{\partial x_3} - e - \frac{\partial e\alpha_1}{\partial x_1} + h\right] \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial^3 s\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} ,$$

$$\frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3} \frac{\partial f\alpha_1}{\partial x_1} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} \frac{\partial f\alpha_2}{\partial x_2} - \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2} \frac{\partial f\alpha_3}{\partial x_3} + \left[- \frac{\partial a\alpha_3}{\partial x_3} - \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3} - \right.$$

$$\left. - \frac{\partial b\alpha_3}{\partial x_3} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \frac{\partial d\alpha_3}{\partial x_3}\right) \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1 \partial x_2} + \left[- \frac{\partial a\alpha_2}{\partial x_2} - \frac{\partial c\alpha_2}{\partial x_2} - \frac{\partial^2 c\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \frac{\partial e\alpha_2}{\partial x_2} - \right.$$

$$\left. - \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3}\right) \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1 \partial x_3} + \left[- \frac{\partial b\alpha_1}{\partial x_1} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} - \frac{\partial c\alpha_1}{\partial x_1} - \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2} + \frac{\partial g\alpha_1}{\partial x_1}\right) \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2 \partial x_3} +$$

$$+ e - a - b - c + d + g - h - \frac{\partial a\alpha_3}{\partial x_3} - \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2 \partial x_3} - \frac{\partial a\alpha_2}{\partial x_2} - \frac{\partial b\alpha_3}{\partial x_3} - \frac{\partial b\alpha_1}{\partial x_1} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1 \partial x_3} -$$

$$\left. - \frac{\partial c\alpha_2}{\partial x_2} - \frac{\partial c\alpha_1}{\partial x_1} - \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1 \partial x_2} + \frac{\partial d\alpha_3}{\partial x_3} + \frac{\partial e\alpha_2}{\partial x_2} - \frac{\partial g\alpha_1}{\partial x_1} - \frac{\partial^2 g\alpha_1\alpha_3}{\partial x_1 \partial x_3}\right) \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} =$$

$$= \frac{\partial^3 s\alpha_1\alpha_2\alpha_3}{\partial x_1 \partial x_2 \partial x_3} . \qquad (6)$$

$$
M' = \begin{bmatrix}
a & b & c & d-a-b & e-a-c & g-b-c & a\cdot b\cdot c-g-e-d\cdot h \\
0 & b\cdot\frac{\partial b_{\alpha_1}}{\partial x_1} & c\cdot\frac{\partial c_{\alpha_1}}{\partial x_1} & 0 & 0 & g\cdot\frac{\partial g_{\alpha_1}}{\partial x_1}-b\cdot\frac{\partial b_{\alpha_1}}{\partial x_1}-c\cdot\frac{\partial c_{\alpha_1}}{\partial x_1} & 0 \\
a\cdot\frac{\partial a_{\alpha_2}}{\partial x_2} & 0 & c\cdot\frac{\partial c_{\alpha_2}}{\partial x_2} & 0 & e\cdot\frac{\partial e_{\alpha_2}}{\partial x_2}-a\cdot\frac{\partial a_{\alpha_2}}{\partial x_2}-c\cdot\frac{\partial c_{\alpha_2}}{\partial x_2} & 0 & 0 \\
a\cdot\frac{\partial a_{\alpha_3}}{x_3} & b\cdot\frac{\partial b_{\alpha_3}}{\partial x_3} & 0 & -a-\frac{\partial a_{\alpha_3}}{\partial x_3}-b-\frac{\partial b_{\alpha_3}}{\partial x_3}\cdot d\cdot\frac{\partial d_{\alpha_3}}{\partial x_3} & 0 & 0 & 0 \\
0 & 0 & c\cdot\frac{\partial c_{\alpha_2}}{\partial x_2}\cdot\frac{\partial c_{\alpha_1}}{\partial x_1}\cdot\frac{\partial^2 c_{\alpha_1\alpha_2}}{\partial x_1\partial x_2} & 0 & 0 & 0 & 0 \\
0 & b\cdot\frac{\partial b_{\alpha_1}}{\partial x_1}\cdot\frac{\partial b_{\alpha_3}}{\partial x_3}\cdot\frac{\partial^2 b_{\alpha_1\alpha_3}}{\partial x_1\partial x_3} & 0 & 0 & 0 & 0 & 0 \\
a\cdot\frac{\partial a_{\alpha_2}}{\partial x_2}\cdot\frac{\partial a_{\alpha_3}}{\partial x_3}\cdot\frac{\partial^2 a_{\alpha_2\alpha_3}}{\partial x_2\partial x_3} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

	s	b	c	$d-b-e$	e	$g-b-c$	$h+b-g-d$
	$s+\frac{\partial s_{\alpha_1}}{\partial x_1}$	$b+\frac{\partial b_{\alpha_1}}{\partial x_1}$	$c+\frac{\partial c_{\alpha_1}}{\partial x_1}$	0	0	$g+\frac{\partial g_{\alpha_1}}{\partial x_1}-b-\frac{\partial b_{\alpha_1}}{\partial x_1}-c-\frac{\partial c_{\alpha_1}}{\partial x_1}$	0
	$s+\frac{\partial s_{\alpha_2}}{\partial x_2}$	0	$c+\frac{\partial c_{\alpha_2}}{\partial x_2}$	$a+\frac{\partial a_{\alpha_2}}{\partial x_2}-c+\frac{\partial c_{\alpha_2}}{\partial x_2}-e-\frac{\partial e_{\alpha_2}}{\partial x_2}$	$e+\frac{\partial e_{\alpha_2}}{\partial x_2}-c-\frac{\partial c_{\alpha_2}}{\partial x_2}$	0	$\frac{\partial e_{\alpha_2}}{\partial x_2}-\frac{\partial c_{\alpha_2}}{\partial x_2}-\frac{\partial a_{\alpha_2}}{\partial x_2}+e-a-c$
$X=\frac{\partial f_{\alpha_1}}{\partial x_1}=$	$s+\frac{\partial s_{\alpha_3}}{\partial x_3}$	$b+\frac{\partial b_{\alpha_3}}{\partial x_3}$	0	$d+\frac{\partial d_{\alpha_3}}{\partial x_3}-b-\frac{\partial b_{\alpha_3}}{\partial x_3}-a-\frac{\partial a_{\alpha_3}}{\partial x_3}$	$a+\frac{\partial a_{\alpha_3}}{\partial x_3}$	0	0
	$s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial^2 s_{\alpha_1\alpha_2}}{\partial x_1\partial x_2}$	0	$c+\frac{\partial c_{\alpha_1}}{\partial x_1}+\frac{\partial c_{\alpha_2}}{\partial x_2}+\frac{\partial^2 c_{\alpha_1\alpha_2}}{\partial x_1\partial x_2}$	0	0	0	0
	$s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_1\alpha_3}}{\partial x_1\partial x_3}$	$b+\frac{\partial b_{\alpha_1}}{\partial x_1}+\frac{\partial b_{\alpha_3}}{\partial x_3}+\frac{\partial^2 b_{\alpha_1\alpha_3}}{\partial x_1\partial x_3}$	0	0	0	0	0
	$s+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_2\alpha_3}}{\partial x_2\partial x_3}$	0	0	0	$a+\frac{\partial a_{\alpha_2}}{\partial x_2}+\frac{\partial a_{\alpha_3}}{\partial x_3}+\frac{\partial^2 a_{\alpha_2\alpha_3}}{\partial x_2\partial x_3}$	0	0

$$Y=\frac{\partial f_{\alpha_2}}{\partial x_2}=\begin{array}{|ccccccc}
a & s & c & d-a & e-a-c & g-c-h+e & h+a-d-c \\
0 & s+\frac{\partial s_{\alpha_1}}{\partial x_1} & c+\frac{\partial c_{\alpha_1}}{\partial x_1} & b+\frac{\partial b_{\alpha_1}}{\partial x_1} & 0 & 0 & g+\frac{\partial g_{\alpha_1}}{\partial x_1}-c-\frac{\partial c_{\alpha_1}}{\partial x_1}-b-\frac{\partial b_{\alpha_1}}{\partial x_1} \\
a+\frac{\partial a_{\alpha_2}}{\partial x_2} & s+\frac{\partial s_{\alpha_2}}{\partial x_2} & c+\frac{\partial c_{\alpha_2}}{\partial x_2} & 0 & e+\frac{\partial e_{\alpha_2}}{\partial x_2}-a-\frac{\partial a_{\alpha_2}}{\partial x_2}-c-\frac{\partial c_{\alpha_2}}{\partial x_2} & 0 & 0 \\
a+\frac{\partial a_{\alpha_3}}{\partial x_3} & s+\frac{\partial s_{\alpha_3}}{\partial x_3} & 0 & d+\frac{\partial d_{\alpha_3}}{\partial x_3}-a-\frac{\partial a_{\alpha_3}}{\partial x_3} & 0 & a+\frac{\partial a_{\alpha_3}}{\partial x_3}+b+\frac{\partial b_{\alpha_3}}{\partial x_3}-d-\frac{\partial d_{\alpha_3}}{\partial x_3} & 0 \\
0 & s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial^2 s_{\alpha_1\alpha_2}}{\partial x_1\partial x_2} & c+\frac{\partial c_{\alpha_1}}{\partial x_1}+\frac{\partial c_{\alpha_2}}{\partial x_2}+\frac{\partial^2 c_{\alpha_1\alpha_2}}{\partial x_1\partial x_2} & 0 & 0 & 0 & 0 \\
0 & s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_1\alpha_3}}{\partial x_1\partial x_3} & 0 & b+\frac{\partial b_{\alpha_1}}{\partial x_1}+\frac{\partial b_{\alpha_3}}{\partial x_3}+\frac{\partial^2 b_{\alpha_1\alpha_3}}{\partial x_1\partial x_3} & 0 & 0 & 0 \\
a+\frac{\partial a_{\alpha_2}}{\partial x_2}+\frac{\partial a_{\alpha_3}}{\partial x_3}+\frac{\partial^2 a_{\alpha_2\alpha_3}}{\partial x_2\partial x_3} & s+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_2\alpha_3}}{\partial x_2\partial x_3} & 0 & 0 & 0 & 0 & 0
\end{array}$$

$Z = \frac{\partial(\alpha_3)}{\partial x_3} =$

a	b	s	$d-a-b$	$d-b-h$	$g\cdot h-d$	$h\cdot b\cdot e-a-d$
0	$b\cdot\frac{\partial b\alpha_1}{\partial x_1}$	$s\cdot\frac{\partial s\alpha_1}{\partial x_1}$	0	0	$g\cdot\frac{\partial g\alpha_1}{\partial x_1}-b-\frac{\partial b\alpha_1}{\partial x_1}$	$c\cdot\frac{\partial c\alpha_1}{\partial x_1}$
$a\cdot\frac{\partial a\alpha_2}{\partial x_2}$	0	$s\cdot\frac{\partial s\alpha_2}{\partial x_2}$	0	$e\cdot\frac{\partial e\alpha_2}{\partial x_2}-a-\frac{\partial a\alpha_2}{\partial x_2}$	$c\cdot\frac{\partial c\alpha_2}{\partial x_2}\cdot a\cdot\frac{\partial a\alpha_2}{\partial x_2}-e-\frac{\partial e\alpha_2}{\partial x_2}$	0
$a\cdot\frac{\partial a\alpha_3}{x_3}$	$b\cdot\frac{\partial b\alpha_3}{\partial x_3}$	$s\cdot\frac{\partial s\alpha_3}{\partial x_3}$	$d\cdot\frac{\partial d\alpha_3}{\partial x_3}-a-\frac{\partial a\alpha_3}{\partial x_3}-b-\frac{\partial b\alpha_3}{\partial x_3}$	0	0	0
0	0	$s\cdot\frac{\partial s\alpha_1}{\partial x_1}\cdot\frac{\partial s\alpha_2}{\partial x_2}\cdot\frac{\partial^2 s\alpha_1\alpha_2}{\partial x_1\partial x_2}$	0	$c\cdot\frac{\partial c\alpha_1}{\partial x_1}\cdot\frac{\partial c\alpha_2}{\partial x_2}\cdot\frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2}$	0	0
0	$b\cdot\frac{\partial b\alpha_1}{\partial x_1}\cdot\frac{\partial b\alpha_3}{\partial x_3}\cdot\frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3}$	$s\cdot\frac{\partial s\alpha_1}{\partial x_1}\cdot\frac{\partial s\alpha_3}{\partial x_3}\cdot\frac{\partial^2 s\alpha_1\alpha_3}{\partial x_1\partial x_3}$	0	0	0	0
$a\cdot\frac{\partial a\alpha_2}{\partial x_2}\cdot\frac{\partial a\alpha_3}{\partial x_3}\cdot\frac{\partial^2 a\alpha_2\alpha_3}{x_2\ x_3}$	0	$s\cdot\frac{\partial s\alpha_2}{\partial x_2}\cdot\frac{\partial s\alpha_3}{\partial x_3}\cdot\frac{\partial^2 s\alpha_2\alpha_3}{\partial x_2\partial x_3}$	0	0	0	0

The system (6) has a solution for $\frac{\partial f\alpha_1}{\partial x_1}$, $\frac{\partial f\alpha_2}{\partial x_2}$ and $\frac{\partial f\alpha_3}{\partial x_3}$ if the rank of the augmented matrix of system (6) is 7.

The augmented matrix of system (6) and the following matrix are equivalent.

The rank of the augmented matrix = rank $\Delta' = 7$, if conditions (5)(u) and (5)(u') are fulfilled.

If condition (5)(u) is fulfilled, then the following determinants give the solutions of the partial derivatives, so the final solution of the functional equation (4) has the following form

$$f(x_1,x_2,x_3) = C - X(x_1,x_2,x_3) - Y(x_1,x_2,x_3) - Z(x_1,x_2,x_3) - \\ - \frac{\partial Y\alpha_3}{\partial x_3} - \frac{\partial X\alpha_2}{\partial x_2} - \frac{\partial X\alpha_3}{\partial x_3} - \frac{\partial^2 X\alpha_2\alpha_3}{\partial x_2 \partial x_3} .$$

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Rezime

GENERALISANE PSEUDO-BULOVE FUNKCIONALNE JEDNAČINE TREĆEG REDA

U radu su dati potrebni i dovoljni uslovi za postojanje rešenja generalisane pseudo-Bulove funkcionalne jednačine trećeg reda kao i sama njena rešenja u eksplicitnom obliku.

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ON A CLASS OF BISEMILATTICES

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Abstract. In the papers [1] and [2] a near-lattice was defined as a bisemilattice (Q,∇,Δ) satisfying the identity:

$$x\Delta(y\nabla z\nabla x) = (x\Delta y)\nabla(x\Delta z)\nabla(x\Delta x).$$

This structure is called here a (Δ,∇)-weak-distributive bisemilattice, and the structure satisfying the dual identity is said here to be a (∇,Δ)-weak-distributive bisemilattice.

In this paper a near-lattice is defined as a bisemilattice which satisfies the identity

$$x\nabla(y\Delta x) = (x\nabla y)\Delta x.$$

Some properties of such structures are proved, and a necessary and sufficient condition for a bisemilattice to be a near-lattice is given.

■

A bisemilattice (Q,∇,Δ) is an algebra with two binary operations such that (Q,∇) and (Q,Δ) are commutative semigroups which satisfy the idempotent laws.

We shall say that a bisemilattice (Q,∇,Δ) is a near-lattice iff for all $x,y\in Q$

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(SM) $x\nabla(y\Delta x) = (x\nabla y)\Delta x$.

(SM) is a self- dual law, and if (Q,∇,Δ) is a near-lattice, then (Q,Δ,∇) is also a near-lattice. Thus, the duality is satisfied in any near-lattice.

Example 1.

$Q = \{a,b,c\}$

∇	a	b	c
a	a	b	a
b	b	b	b
c	a	b	c

Δ	a	b	c
a	b	b	b
b	b	b	b
c	b	b	c

The bisemilattice (Q,∇,Δ) is not a near-lattice, since $a\nabla(c\Delta a) = b \neq a = (a\nabla c)\Delta a$.

In a bisemilattice (Q,∇,Δ) (SM) follows from the identity $x\Delta(y\nabla z\nabla x) = (x\Delta y)\nabla(x\Delta z)\nabla(x\Delta x)$ (replacing z by x). It also follows from the dual identity. Hence:

Proposition 1. [1][2] Every (Δ,∇)-weak-distributive bisemilattice (as a (∇,Δ)-weak-distributive bisemilattice) is a near-lattice. ■

Example 2.

$Q = \{a,b,c\}$

∇	a	b	c
a	a	b	c
b	b	b	b
c	c	b	c

Δ	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

The bisemilattice (Q,∇,Δ) is a (Δ,∇) -weak-distributive bisemilattice, and so it is a near-lattice, but since $c\nabla(a\Delta b\Delta c)=c \neq b=(c\nabla a)\Delta(c\nabla b)\Delta c$, it is not a (∇,Δ)-weak-distributive bisemilattice.

If (Q,∇,Δ) is a near-lattice, we define partial orders on the set Q by:

$$a \le_{\nabla} b \quad \text{iff} \quad a\nabla b = b \qquad \text{and}$$
$$a \le_{\Delta} b \quad \text{iff} \quad a\Delta b = a.$$

We represent a near-lattice (Q,∇,Δ) by Hasse diagrams of the semilattices (Q,∇) and (Q,Δ). If $a \le_{\nabla} b$, then in a Hasse diagram of (Q,∇), we draw b above a, and if $a \le_{\Delta} b$, then we do the same in a Hasse diagram of (Q,Δ).

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[2] *This proposition gave the initial idea for this paper.*

Example 3.

$Q = \{a, b, c, d, e\}$

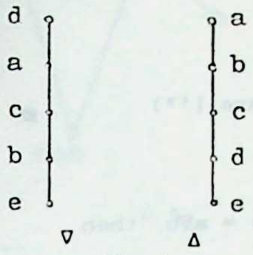


Fig. 1

Since $b\Delta(c\nabla a\nabla b) \neq (b\Delta c)\nabla(b\Delta a)\nabla b$, and $c\nabla(e\Delta d\Delta c) \neq (c\nabla e)\Delta(c\nabla d)\Delta c$, the bisemilattice (Q,∇,Δ) is neither a (∇,Δ), nor a (Δ,∇)-weak-distributive bisemilattice, but it is a near-lattice.

In a near-lattice (Q,∇,Δ), $a\nabla b = b$ implies

$$a\nabla(a\Delta b) = a\Delta(a\nabla b) = a\Delta b, \text{ and } b\nabla(a\Delta b) = (b\nabla a)\Delta b = b.$$

Thus, we have:

Lemma 2. Let (Q,∇,Δ) be a near-lattice. If for $a, b \in Q$, $a \leq_\nabla b$, then $a\nabla(a\Delta b) = a\Delta b$ and $b\nabla(a\Delta b) = b$. ■

Dually, we have:

Lemma 2'. Let (Q,∇,Δ) be a near-lattice. If for $a, b \in Q$, $b \leq_\Delta a$, then $a\Delta(a\nabla b) = a\nabla b$ and $b\Delta(a\nabla b) = b$. ■

Corollary 3. If (Q,∇,Δ) is a near-lattice and $b \leq_\nabla a$, then $b \leq_\nabla a\Delta b \leq_\nabla a$. ■

Corollary 3'. If (Q,∇,Δ) is a near-lattice and $b \leq_\Delta a$, then $b \leq_\Delta a\nabla b \leq_\Delta a$. ■

Corollary 4. If (Q,∇,Δ) is a near-lattice, then

(a) $a \leq_\nabla a\Delta(a\nabla b) \leq_\nabla a\nabla b$,

(b) $b \leq_\nabla b\Delta(a\nabla b) \leq_\nabla a\nabla b$,

(c) $a\Delta b \leq_\Delta a\nabla(a\Delta b) \leq_\Delta a$,

(d) $a\Delta b \leq_\Delta b\nabla(a\Delta b) \leq_\Delta b$. ■

Proposition 5. Let (Q,∇,Δ) be a near-lattice. If for $a, b \in Q$

(*) $a\Delta(b\nabla a) = b\Delta(a\nabla b)$, then

(**) $a\Delta b = a\nabla b = a\Delta(b\nabla a) = b\Delta(a\nabla b) = (a\Delta b)\nabla a = (b\Delta a)\nabla b$.

Proof. Let a and b be comparable in any of two semilattices, for instance let $a\nabla b = b$. Then, using (*), $a\Delta b = b$, and hence we have (**).

Let a and b be incomparable. By Corollaries 3 and 3', since $a\Delta b <_{\Delta} a$ and $a\Delta b <_{\Delta} b$,

$a\Delta b \leq_{\Delta} (a\Delta b)\nabla b \leq_{\Delta} b$.

Since $(a\Delta b)\nabla a = (a\Delta b)\nabla b$, we have

$(a\Delta b)\nabla a = (a\Delta b)\nabla b = a\Delta b$. Dually,

$(a\nabla b)\Delta a = (a\nabla b)\Delta b = a\nabla b$, hence, we have (**). ■

Lemma 6. If (Q,∇,Δ) is a near-lattice, then

(a) For all $a,b\in Q$, if $a\Delta b \leq_{\nabla} a$ and $b\nabla(a\Delta b) = a\nabla b$, then $a\Delta b = a$.

(b) For all $a,b\in Q$, if $(a\Delta b)\nabla b = a\nabla b$ and $a \leq_{\nabla} (a\Delta b)\nabla a \leq_{\nabla} a\nabla b$, then $(a\Delta b)\nabla a = a\Delta b$.

(c) For all $a,b\in Q$ if $a\nabla b = (a\Delta b)\nabla a = (a\Delta b)\nabla b$, then $a\nabla b = a\Delta b$.

Proof.

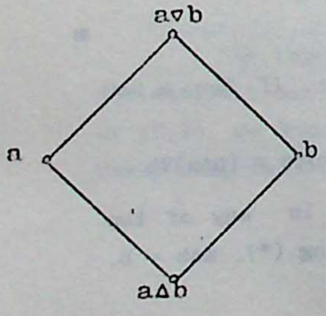
(a) If $a\Delta b \leq_{\Delta} a$, then $a = (a\Delta b)\nabla a = a\Delta(b\nabla a)$. Since $a\nabla b = b\nabla(a\Delta b) = (b\nabla a)\Delta b$, we have that

$$a\Delta b = (a\Delta(b\nabla a))\Delta b = a\Delta((b\nabla a)\Delta b) = a\Delta(a\nabla b) = a.$$

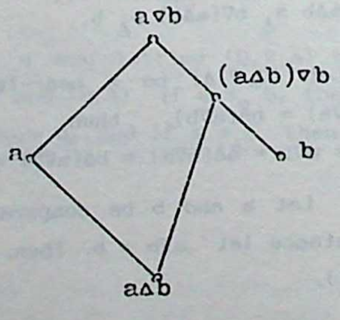
(b) From $a\nabla b = (a\Delta b)\nabla b = b\Delta(a\nabla b)$, it follows that $a\nabla b \leq_{\Delta} b$. By Corollary 4(c), since $a\Delta b \leq_{\Delta} a\Delta(a\nabla b) \leq_{\Delta} a$, it follows that $a\Delta b \leq_{\Delta} a\Delta(a\nabla b) \leq_{\Delta} a\Delta b$, hence $(a\Delta b)\nabla a = a\Delta(a\nabla b) = a\Delta b$.

(c) From $a\nabla b = (a\Delta b)\nabla a = a\Delta(b\nabla a)$, and $a\nabla b = (a\Delta b)\nabla b = b\Delta(a\nabla b)$, it follows that $a\nabla b \leq_{\Delta} a$, and $a\nabla b \leq_{\Delta} b$, hence $a\nabla b \leq_{\Delta} a\Delta b$. By Corollary 4(c) $a\Delta b \leq_{\Delta} a\nabla b$, hence $a\nabla b = a\Delta b$. ■

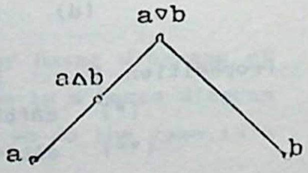
Lemma 7. Let (Q,∇,Δ) be a near-lattice, and $a,b\in Q$. If a and b are incomparable under $\leq_{\nabla}$, then there are 6 up to the isomorphism different subsemilattices generated by a, b and $a\Delta b$ in the semilattice (Q,∇). (See the diagrams in Figures 2.1-2.6).



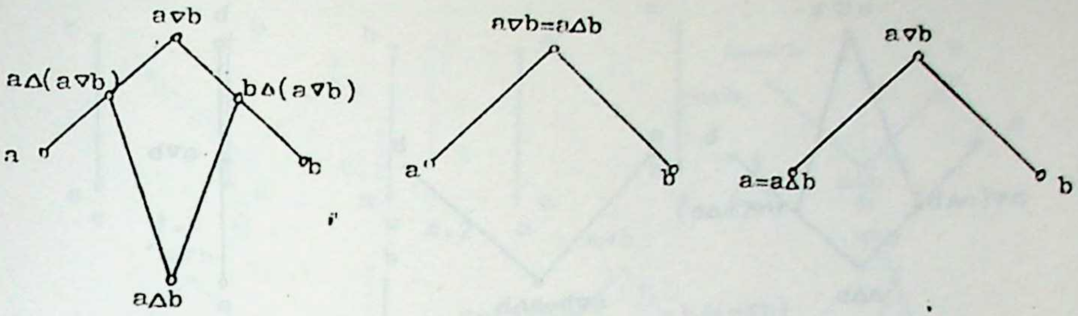
2.1



2.2



2.3



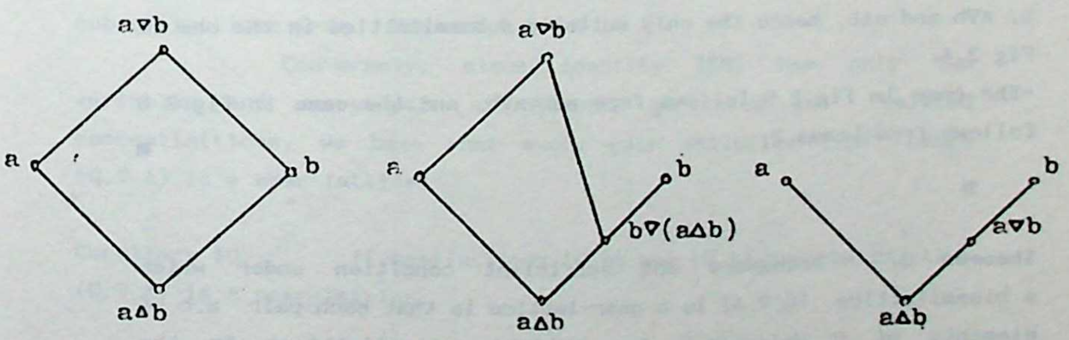
2.4 2.5 2.6

Fig.2

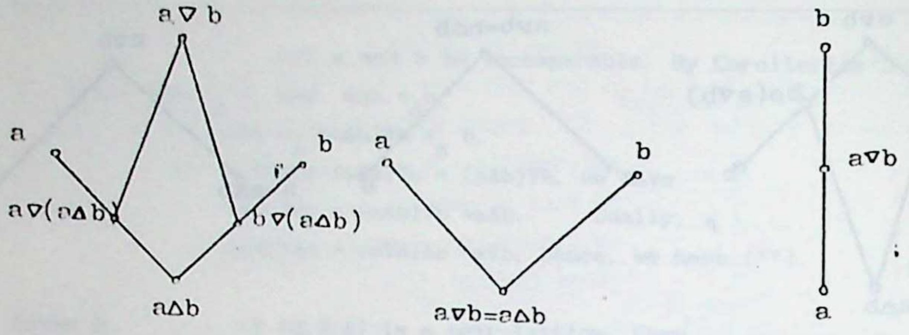
Proof.

By Corollaries 4(a) and 4(b), $a \le_{\nabla} (a\Delta b)\nabla a \le_{\nabla} a\nabla b$, and $b \le_{\nabla} b\nabla(a\Delta b) \le_{\nabla} a\nabla b$. There are nine cases: $a\nabla(a\Delta b) = a$, $a <_{\nabla} a\nabla(a\Delta b) <_{\nabla} a\nabla b$ and $a\nabla(a\Delta b) = a\nabla b$, combined with cases $b\nabla(a\Delta b) =$ $= b$, $b <_{\nabla} b\nabla(a\Delta b) <_{\nabla} a\nabla b$ and $b\nabla(a\Delta b) = a\nabla b$. Also, we differ cases when $a\nabla(a\Delta b) = a\Delta b$, and $a\nabla(a\Delta b) \neq a\Delta b$. Using Lemma 6 we have that subsemilattices represented in Figures 2.1-2.6 are the only, up to the isomorphism, subsemilattices generated by a,b and $a\Delta b$ in the semilattice (Q,∇). ■

Lemma 8. Let (Q,∇,Δ) be a near-lattice and $a,b\in Q$. If a,b and $a\Delta b$ generate subsemilattices of the semilattice (Q,∇) represented in Figures 2.1-2.6, respectively, then a,b and $a\nabla b$ generate subsemilattices of the semilattice (Q,Δ) represented in Figures 3.1-3.6, respectively.



3.1 3.2 3.3



3.4 3.5 3.6

Fig.3

Proof.

-The case in Fig.2.1

From $a = a\nabla(a\Delta b) = a\Delta(a\nabla b)$ and $b = b\nabla(a\Delta b) = b\Delta(a\nabla b)$, it follows that $a \le_\Delta a\nabla b$ and $b \le_\Delta a\nabla b$. Since a and b are incomparable under $\le_\Delta$ (otherwise we would have $a\Delta b = a$ or $a\Delta b = b$, which is not case), the only suitable subsemilattice is the one in Fig.3.1.

-The case in Fig.2.2

$a = (a\Delta b)\nabla a = a\Delta(b\nabla a)$, and $(a\Delta b)\nabla b = (a\nabla b)\Delta b$, and the element $(a\Delta b)\nabla b$ differs from $a\Delta b$ and from b, hence the only suitable subsemilattice is the one in Fig.3.2.

-The case in Fig.2.3

From $a\Delta b = a\nabla(a\Delta b) = a\Delta(a\nabla b) = a\Delta b\Delta(a\nabla b)$ it follows that $a\Delta b <_\Delta a\nabla b$, and from $a\nabla b = (a\Delta b)\nabla b = b\Delta(a\nabla b)$, we have that $a\nabla b <_\Delta b$, hence the only suitable subsemilattice is the one in Fig. 3.3.

-The case in Fig.2.4

Elements $(a\Delta b)\nabla a = a\Delta(b\nabla a)$ and $(b\Delta a)\nabla b = b\Delta(a\nabla b)$ differ from a, b, $a\nabla b$ and $a\Delta b$, hence the only suitable subsemilattice is the one in Fig.3.4.

-The case in Fig.2.5 follows from $a\Delta b=a\nabla b$, and the case in Fig.2.6 follows from Lemma 2. ■

Theorem 9. Necessary and sufficient condition under which a bisemilattice (Q,∇,Δ) is a near-lattice is that each pair a, b of elements of Q determines two related subsemilattices in the semilattices (Q,∇) and (Q,Δ), the pairs of related subsemilattices being represented in figures 4.1-4.9.

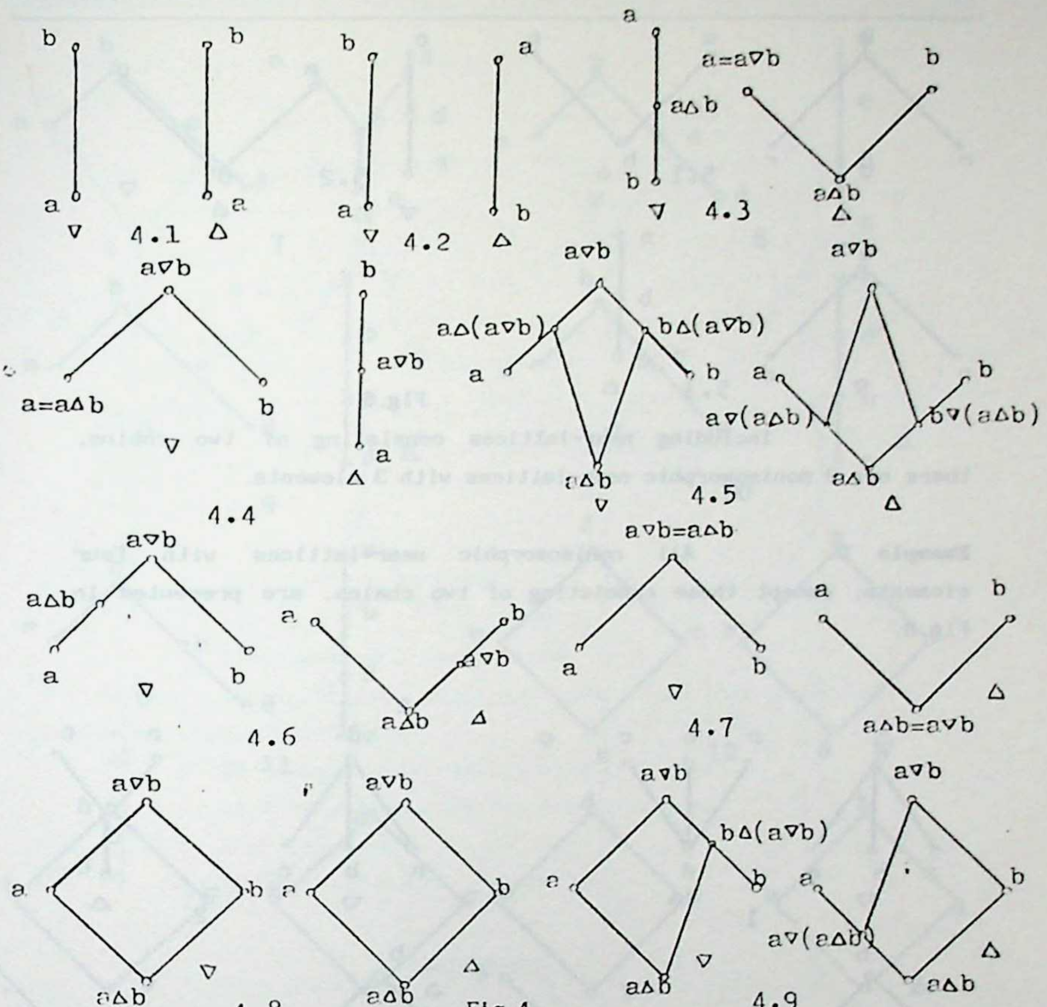


Fig. 4

Proof. By Lemmas 2, 6, 7 and 8 we have that if (Q,∇,Δ) is a near-lattice, then all $a,b\in Q$ generate a pair of related subsemilattices from Figures 4.1-4.9.

Conversely, since identity (SM) has only two variables and every pair of elements generates two related subsemilattices, we have that every pair satisfies (SM), hence (Q,∇,Δ) is a near-lattice. ■

Corollary 10. If semilattices (Q,∇) and (Q,Δ) are chains, then (Q,∇,Δ) is a near-lattice.

In Examples 4 and 5 we give all, up to the isomorphism, different near-lattices with 3 and 4 elements.

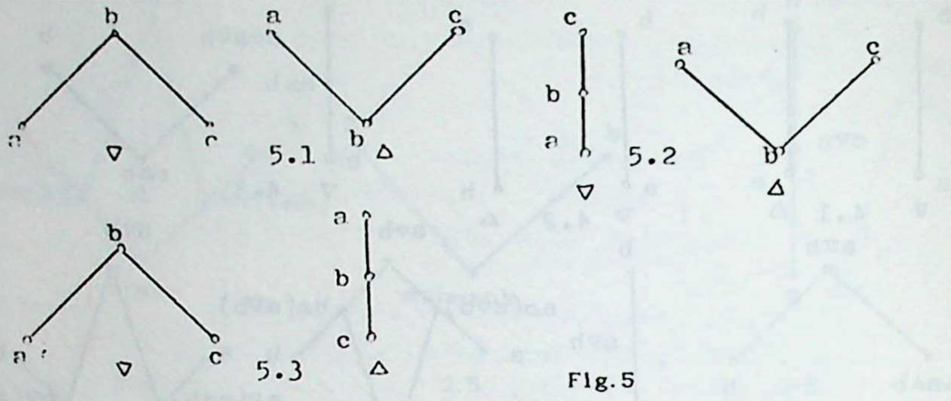
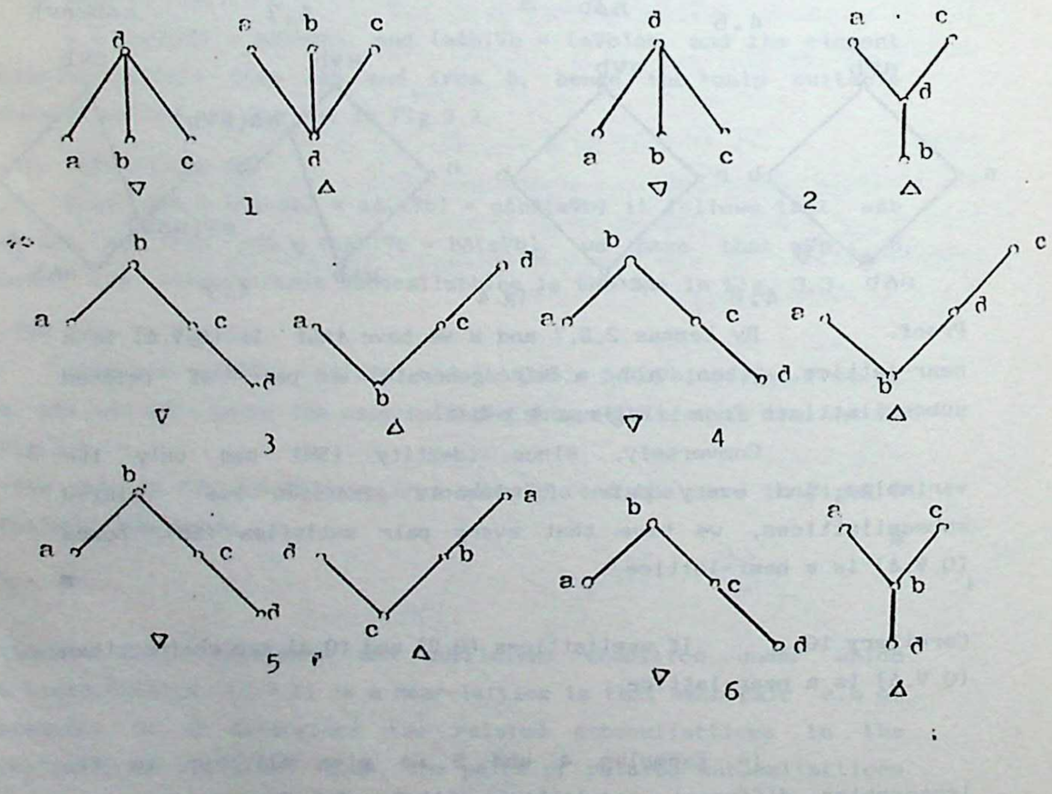
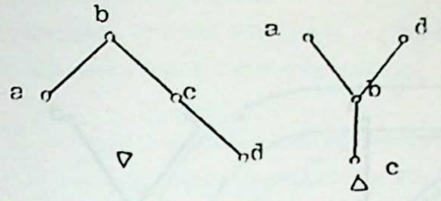


Fig.5

Including near-lattices consisting of two chains, there are 9 nonisomorphic near-lattices with 3 elements.

Example 5. All nonisomorphic near-lattices with four elements, except these consisting of two chains, are presented in Fig.6.





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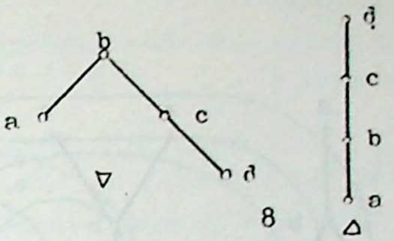

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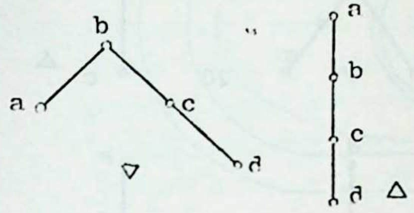

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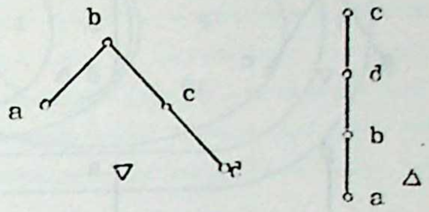

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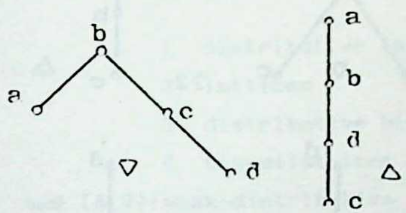

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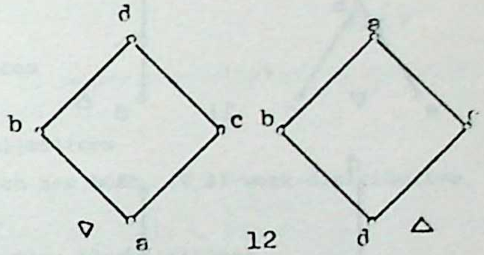

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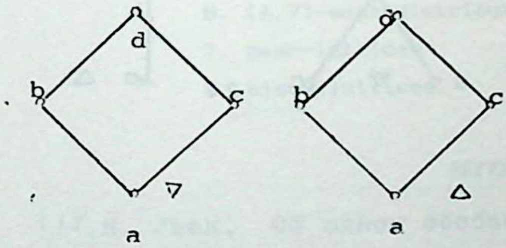

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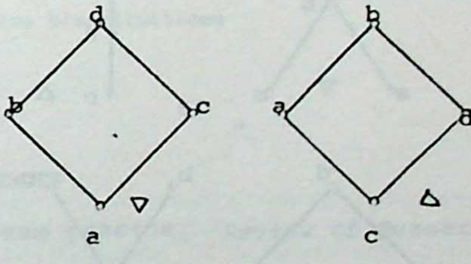

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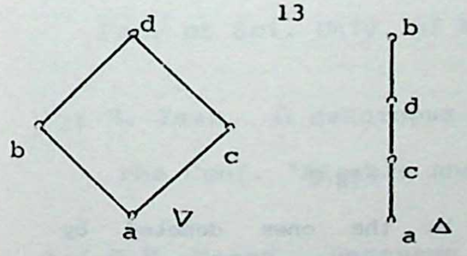

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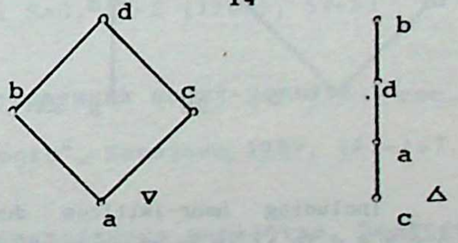

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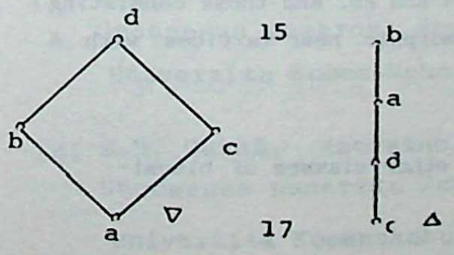

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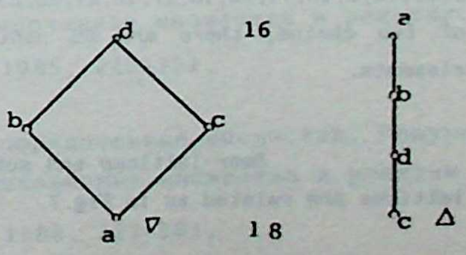

18

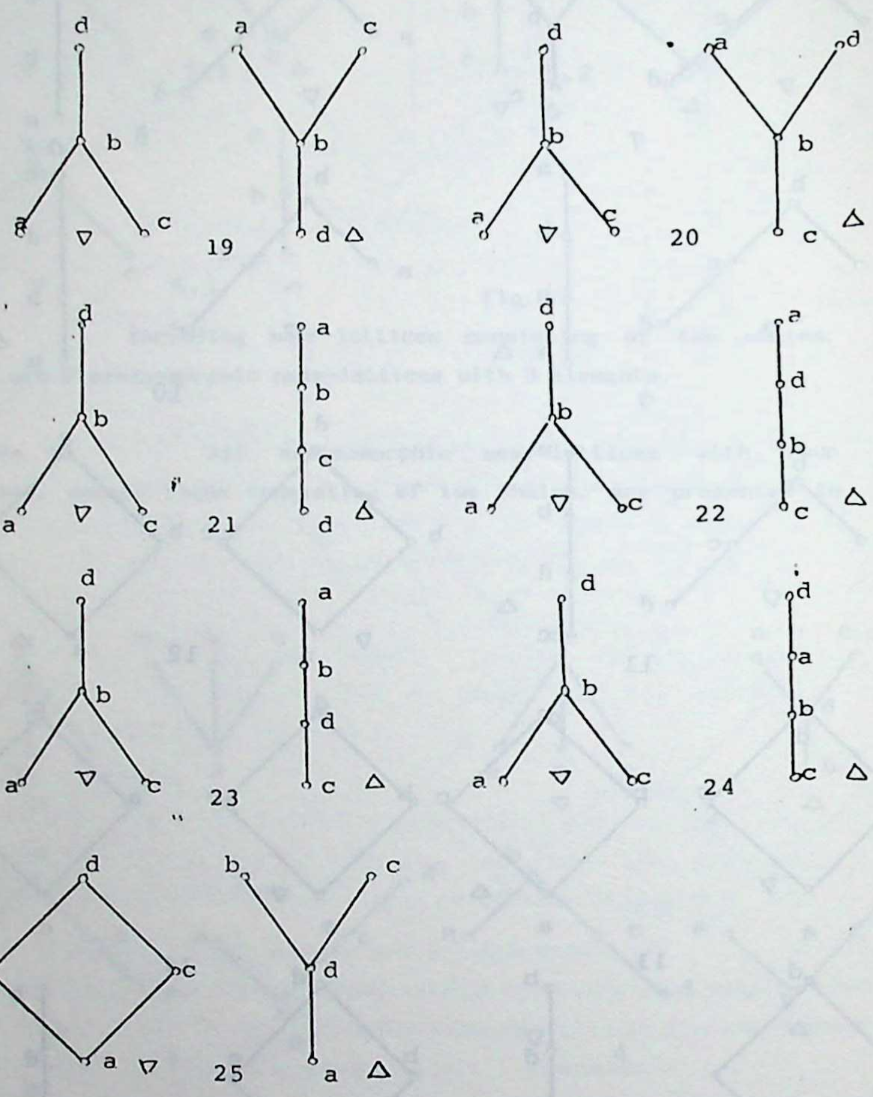

Fig.6

Including near-lattices dual to the ones denoted by 2,6,7,8,9,10,11,15,16,17,18,21,22,23,24 and 25, and those consisting of two chains, there are 65 nonisomorphic near-lattices with 4 elements.

Near-lattices and some other classes of bisemilattices are related as in Fig.7.

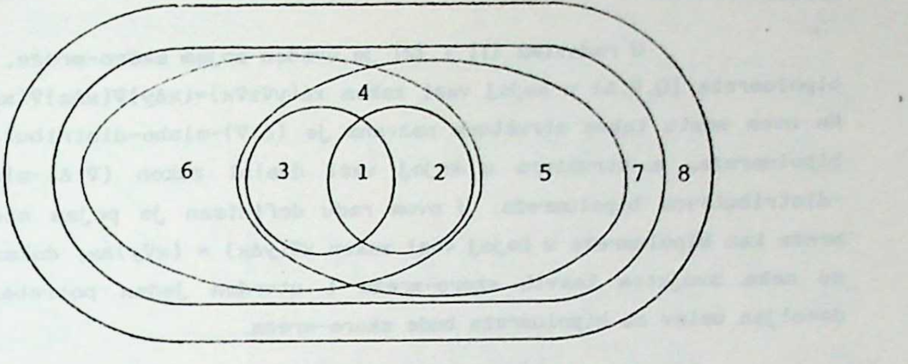

Fig.7

1. distributive lattices
2. lattices
3. distributive bisemilattices
4. bisemilattices which are both, (∇,Δ)-weak-distributive, and (Δ,∇)-weak-distributive
5. (∇,Δ)-weak-distributive bisemilattices
6. (Δ,∇)-weak-distributive bisemilattices
7. near-lattices
8. bisemilattices.

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O JEDNOJ KLASI BIPOLUMREŽA

REZIME

U radovima [1] i [2] je uveden pojam skoro-mreže, kao bipolumreže (Q,∇,Δ) u kojoj važi zakon $x\Delta(y\nabla z\nabla x)=(x\Delta y)\nabla(x\Delta z)\nabla(x\Delta x)$. Na ovom mestu takva struktura nazvana je (Δ,∇)-slabo-distributivna bipolumreža, a struktura u kojoj važi dualni zakon (∇,Δ)-slabo-distributivna bipolumreža. U ovom radu definisan je pojam skoro-mreže kao bipolumreže u kojoj važi zakon $x\nabla(y\Delta x) = (x\nabla y)\Delta x$, dokazana su neka svojstva takvih skoro-mreža i utvrđen jedan potreban i dovoljan uslov da bipolumreža bude skoro-mreža.

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n-РЕШЕТКИ

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PEЗЮМЕ

В статье определяются A- и R-n-решетки как одно обобщение A- и R-решеток. Притом, в R-n-решетке точные верхние и точные нижние граньи определяются как (n-2)-арные операции. Устанавливается соотношение между A- и R-n-решетками и описывается изображение n-решеток с помощю решеток. Наконец, доказывается, что в конечных n-решетках существуют (однозначно определенные) (n-2)-арные операции O и E, являющиеся {1,n}-нейтральными операциями [1],в том же порядке, n-группоидов $(S,\cap)$ и $(S,\cup)$,и {1,n}-нулевими операциями, в том же порядке, n-группоидов $(S,\cap)$ и $(S,\cup)$.

*

Определение 1. Пусть $\cup$ и $\cap$ n-арные операции в множестве S и $n \in N \setminus \{1\}$. Объект $(S,\cup,\cap)$ назовем A-n-решеткой тогда и только тогда, когда имеют место равенства:

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* AMS *Mathematics Subject Classification (1980): 20N15.*
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$n\Lambda L\breve{1}$ $\quad \cup(x,a_1^{n-2},x) = x^{1)}$,

$n\Lambda L\hat{1}$ $\quad \cap(x,a_1^{n-2},x) = x$,

$n\Lambda L\breve{2}$ $\quad \cup(x,a_1^{n-2},y) = \cup(y,a_1^{n-2},x)$,

$n\Lambda L\hat{2}$ $\quad \cap(x,a^{n-2},y) = \cap(y,a_1^{n-2},x)$,

$n\Lambda L\breve{3}$ $\quad \cup(\cup(x,a_1^{n-2},y),a_1^{n-2},z) = \cup(x,a_1^{n-2},\cup(y,a_1^{n-2},z))$,

$n\Lambda L\hat{3}$ $\quad \cap(\cap(x,a_1^{n-2},y),a_1^{n-2},z) = \cap(x,a_1^{n-2},\cap(y,a_1^{n-2},z))$, и

$n\Lambda L4$ $\quad \cup(x,a_1^{n-2},\cap(x,a_1^{n-2},y)) = x$,

$n\Lambda L4$ $\quad \cap(x,a^{n-2},\cup(x,a^{n-2},y)) = x$

для любых $x,y,z,a_1^{n-2} \in S$.

Если $n = 2$, то речь идет об определении A-решетки.[1]

УТВЕРЖДЕНИЕ 1. Пусть $(S,\cup,\cap)$ A-n-решетка. Пусть, далее,

$$(1) \qquad (x,a_1^{n-2},y) \in \leq \quad \overset{\text{деф}}{\Longleftrightarrow} \quad \cup(x,a_1^{n-2},y) = y$$

для любых $x,y,a_1^{n-2} \in S$. Тогда имеют место формулы:

$nRL1$ $\quad (\forall a_i \in S)_1^{n-2}(\forall x \in S)(x,a_1^{n-2},x) \in \leq^{1)}$;

$nRL2$ $\quad (\forall a_i \in S)_1^{n-2}(\forall x \in S)(\forall y \in S)((x,a_1^{n-2},y) \in \leq \wedge$
$\quad \wedge\ (y,a_1^{n-2},x) \in \leq \Rightarrow x = y)$; и

$nRL3$ $\quad (\forall a_i \in S)_1^{n-2}(\forall x \in S)(\forall y \in S)(\forall z \in S)((x,a_1^{n-2},y) \in \leq \wedge$

---

[1] Последовательность $g_p\ \ldots, g_q$ (где $p,\ldots,q$ последовательность натуральных чисел следующих друг за другом) для краткости обозначаем через g_p^q; в частности: $g_p^p = g_p$. Подобно, последовательность $\underbrace{a,\ldots,a}_{p}$ обозначаем через $\overset{p}{a}$. Далее, под g_p^{p-1}, $p \in N$, и под $\overset{0}{g}$ понимаем пустую последовательность (отсуствие последовательности). Притом, если речь идет о, например, g_1^3, e_1^0 (или $g_1^3, \overset{0}{e}$), то g_1^3, e_1^0 (или $g_1^3, \overset{0}{e}$) считаем последовательностью g_1^3, и если реч идет о e_1^0 (или $\overset{0}{e}$) в некоторых случаях будем обозначать через $\varnothing$; например $F(a_1^0)$ будем обозначать через $F(\varnothing)$.

$$\wedge\ (y,a_1^{n-2},z) \in \leq \Rightarrow (x,a^{n-2},z) \in \leq).$$

ДОКАЗАТЕЛЬСТВО.

а) Учитывая (1), ввиду $nAL\bar{1}$, находим, что имеет место формула под $nRL1$.

б) Пусть x,a_1^{n-2} и y любые элементы множества S удовлетворяющие условию

$$(x,a_1^{n-2},y) \in \leq \wedge (y,a_1^{n-2},x) \in \leq,$$

т.е., ввиду (1), условию

$$U(x,a_1^{n-2},y) = y \wedge U(y,a_1^{n-2},x) = x$$

Отсюда, ввиду $nAL\bar{2}$, находим, что имеет место равенство $x = y$.

ц) Пусть x,a_1^{n-2},y и z любые элементы множества S удовлетворяющие условию

$$(x,a_1^{n-2},y) \in \leq \wedge (y,a_1^{n-2},z) \in \leq,$$

т.е., ввиду (1), условию

$$U(x,a_1^{n-2},y) = y \wedge U(y,a_1^{n-2},z) = z.$$

Отсюда, учитывая $nAL\bar{3}$, находим, что имеет место следующая цепь равентств:

$$\begin{aligned} z &= U(U(x,a_1^{n-2},y),a_1^{n-2},z) = \\ &= U(x,a_1^{n-2},U(y,a_1^{n-2},z)) = \\ &= U(x,a_1^{n-2},z), \end{aligned}$$

т.е. равенство

$$U(x,a_1^{n-2},z) = z.$$

Таким образом, ввиду (1), имеет место и формула под $nRL3$.

Если $n = 2$, то $\leq$ из утверждения I является (бинарным) PAT отношением.

УТВЕРЖДЕНИЕ 2. Если $(S,\cup,\cap)$ A-n-решетка, то имеет место эквивалентность

$(\bar{\text{I}})$ $\qquad \cap(x,a_1^{n-2},y) = x \Leftrightarrow \cup(x,a_1^{n-2},y) = y$ 2)

для любых $x,y,a_1^{n-2} \in S$.

ДОКАЗАТЕЛЬСТВО.

а) Учитывая nAL4, находим, что имеет место следующая цепь импликаций

$$\cup(x,a_1^{n-2},y) = y \Rightarrow \cap(x,a_1^{n-2},\cup(x,a_1^{n-2},y)) = $$
$$= \cap(x,a_1^{n-2},y) \Rightarrow x = \cap(x,a_1^{n-2},y)$$

для любых $x,y,a_1^{n-2} \in S$. Отсюда находим, что имеет место импликация

$$\cup(x,a_1^{n-2},y) = y \Rightarrow \cap(x,a_1^{n-2},y) = x$$

для любых $x,y,a_1^{n-2} \in S$.

б) Подобно, учитывая nAL4, nAL$\breve{3}$ и nAL$\hat{3}$, находим, что имеет место следующая цепь импликаций

$$\cap(x,a_1^{n-2},y) = x \Rightarrow \cup(y,a_1^{n-2},\cap(x,a_1^{n-2},y)) =$$
$$= \cup(y,a_1^{n-2},x) \Rightarrow y = \cup(y,a_1^{n-2},x)$$
$$\Rightarrow \cup(x,a_1^{n-2},y) = y$$

для любых $x,y,a_1^{n-2} \in S$. Отсюда находим, что имеет место импликация

$$\cap(x,a_1^{n-2},y) = x \Rightarrow \cup(x,a_1^{n-2},y) = y$$

---

2) см. (1).

для любых $x,y,a_1^{n-2} \in S$.

* *

Пусть

$$\overrightarrow{S^{n-2}S} \stackrel{\text{деф}}{=\!=} \{f \mid f : S^{n-2} \to S\}.$$

Если $n = 2$, то $\overrightarrow{S^{n-2}S}$ является множеством всех нульарных операций в множестве S, именно:

$$\overrightarrow{S^{0}S} = \{\{(\emptyset,a)\} \mid a \in S\}.$$

Так как нульарная операция

$$f = \{(\emptyset,a)\}$$

однозначно определена элементом $a \in S$ и $f(\emptyset) = a$, то $\overrightarrow{S^{0}S}$ будем обозначат и через S, т.е. то будем считать, что

(0) $\overrightarrow{S^{0}S} = S.$

ОПРЕДЕЛЕНИЕ 2_1. Пусть $\le$ n-арное отношение, $n \in N\setminus\{1\}$, в множестве S удовлетворяющее условыям nRL1-nRL3. Пусть, далее, $D \subseteq S$. Тогда:

1° $M \in \overrightarrow{S^{n-2}S}$ назовем верхней гранью множества D в $(S,\le)$ тогда и только тогда, когда имеет место условые:

(B) $(\forall a_i \in S)^{n-2}(\forall x \in S)(x \in D \Rightarrow (x,a_1^{n-2},M(a_1^{n-2})) \in \le)$; и

2° $m \in \overrightarrow{S^{n-2}S}$ назовем нижней гранью множества D в $(S,\le)$ тогда и только тогда, когда имеет место условые:

(H) $(\forall a_i \in S)_1^{n-2}(\forall x \in S)(x \in D \Rightarrow (m(a_1^{n-2}),a_1^{n-2},x) \in \le)$.

Притом, множество всех верхних граней и множество всех нижних граней множества D в $(S,\le)$, в том же порядке, обозначим через

$\mathcal{G}(D/\le)$ и $\mathcal{D}(D/\le)$.

Если $n = 2$, то, ввиду (C), речь идет об определении верхней и нижней грани множества $D \subseteq S$ в $(S,\le)$.

УТВРЕЖДЕНИЕ 3. Пусть $\le$ n-арное отношение, $n \in N \setminus \{1\}$, в множестве S удовлетворяющее условиям nRL1-nRL3. Пусть, далее, $D \subseteq S$. Тогда:

$\bar{1}$ существует не больше чем одна $M_D \in \overrightarrow{S^{n-2}S}$ удовлетворяющая условию

$$\text{(M)} \qquad M_D \in \mathcal{G}(D/\le) \wedge (\forall a_i \in S)_1^{n-2}(\forall X \in \overrightarrow{S^{n-2}S})(X \in \mathcal{G}(D/\le)) \Rightarrow$$

$$\Rightarrow (M_D(a_1^{n-2}), a_1^{n-2}, X(a_1^{n-2})) \in \le); \text{ и}$$

$\bar{2}$ существует не больше чем одна $m_D \in \overrightarrow{S^{n-2}S}$ удовлетворяющая условию

$$\text{(m)} \qquad m_D \in \mathcal{D}(D/\le) \wedge (\forall a_i \in S)_1^{n-2}(\forall X \in \overrightarrow{S^{n-2}S})(X \in \mathcal{D}(D/\le)) \Rightarrow$$

$$\Rightarrow (X(a_1^{n-2}), a_1^{n-2}, m_D(a_1^{n-2})) \in \le).$$

ДОКАЗАТЕЛЬСТВО.

Пусть M_D и $\bar{M}_D$ удовлетворяют условию под (M). Тогда имеет место конъюнкция

$$(M_D(a_1^{n-2}), a_1^{n-2}, \bar{M}_D(a_1^{n-2})) \in \le \wedge (\bar{M}_D(a_1^{n-2}), a_1^{n-2}, M_D(a_1^{n-2})) \in \le$$

для любых $a_1^{n-2} \in S$. Отсюда, ввиду nRL2, находим, что имеет место равенство

$$\bar{M}_D(a_1^{n-2}) = M_D(a_1^{n-2})$$

для каждых $a_1^{n-2} \in S$.

Таким же способом доказывается, что имеет место утверждение под $\bar{2}$.

ОПРЕДЕЛЕНИЕ 2_2. Пусть $\le$ n-арное, $n \in N \setminus \{1\}$, отношение в множестве S удовлетворяющее условыям nRL1-nRL3. Пусть, далее,

$D \subseteq S$. Тогда

1' $M_D \in \overrightarrow{S^{n-2}S}$ назовем точной верхней гранью множества D тогда и только тогда, когда M_D удовлетворяет условию (M); и

2' $m_D \in \overrightarrow{S^{n-2}S}$ назовем точной нижней гранью множества D тогда и только тогда, когда m_D удовлетворяет условию (m). Притом M_D и m_D в том же порядке, обозначим через $\sup_{\leq} D$ ($\sup D$) и $\inf_{\leq} D$ ($\inf D$).

ОПРЕДЕЛЕНИЕ 2_3. Пусть $\leq$ n-арное, $n \in N \setminus \{1\}$, отношение в множестве S. Объект $(S,\leq)$ назовем R-n-решеткой тогда и только тогда, когда $\leq$ удовлетворяет формулам nRL1-nRL3 и условию

nRL4 $(\forall x \in S)(\forall y \in S)(\exists M \in \overrightarrow{S^{n-2}S})(\exists m \in \overrightarrow{S^{n-2}S})(M = \sup_{\leq}\{x,y\} \wedge m = \inf_{\leq}\{x,y\})$.

ТЕОРЕМА 4. Пусть $(S,\cup,\cap)$ Λ-n-решетка. Пусть, далее,

(1) $(x,a_1^{n-2},y) \in \leq \overset{\text{деф}}{\Longleftrightarrow} \cup(x,a_1^{n-2},y) = y$ [3].

Тогда $(S,\leq)$ является R-n-решеткой.

ДОКАЗАТЕЛЬСТВО.

Ввиду утверждения 1, $\leq$ удовлетворяет условиям nRL1-nRL3.

Пусть, далее, (a_1^{n-2}) любой элемент множества S^{n-2}. Притом, пусть:

(α) $\underset{\smile}{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} \cup(x,a_1^{n-2},y)$;

(β) $\overset{\frown}{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=\!=} \cap(x,a_1^{n-2},y)$; и

(γ) $(x,y) \in \leq_{(a_1^{n-2})} \overset{\text{деф}}{\Longleftrightarrow} (x,a_1^{n-2},y) \in \leq$

для каждого $(x,y) \in S^2$.

Учитывая (α), (β) и определение 1, находим, что для каждого $(a_1^{n-2}) \in S^{n-2}$ объект

---

[3] $\Leftrightarrow \cap(x,a_1^{n-2},y) = x$; утвреждение 2.

$$(S, \underparen{(a_1^{n-2})}, \overparen{(a_1^{n-2})})$$

является Λ-решеткой. Далее, учитывая (γ), (1) и (α) (или (β)), находим, что

$$(S, \le_{(a_1^{n-2})})$$

R-решетка соответствующая Λ-решетке

$$(S, \underparen{(a_1^{n-2})}, \overparen{(a_1^{n-2})})$$

для каждого $(a_1^{n-2}) \in S^{n-2}$. Отсюда находим, что

$$M_{\{x,y\}}(a_1^{n-2}) = \sup_{\le_{(a_1^{n-2})}}\{x,y\} = \underparen{(a_1^{n-2})}(x,y) \text{ и}$$

$$m_{\{x,y\}}(a_1^{n-2}) = \inf_{\le_{(a_1^{n-2})}}\{x,y\} = \overparen{(a_1^{n-2})}(x,y)$$

для каждого $(x,y) \subset S^2$ и каждого $(a_1^{n-2}) \in S^{n-2}$. Притом, если множество всех верхних и множество всех нижних граньи $z \in S$ множества $\{x,y\}$ в $(S, \le_{(a_1^{n-2})})$, в том же порядке, обозначим через

$$G(\{x,y\}/\le_{(a_1^{n-2})}) \text{ и } \mathcal{D}(\{x,y\}/\le_{(a_1^{n-2})}),$$

то имеют место импликации

$$z \in G(\{x,y\}/\le_{(a_1^{n-2})}) \Rightarrow (M_{\{x,y\}}(a_1^{n-2}), z) \in \le_{(a_1^{n-2})} \text{ и}$$

$$z \in \mathcal{D}(\{x,y\}/\le_{(a_1^{n-2})}) \Rightarrow (z, m_{\{x,y\}}(a_1^{n-2})) \in \le_{(a_1^{n-2})}$$

для всех $z \in S$, каждого $(x,y) \in S^2$ и каждого $(a_1^{n-2}) \in S^{n-2}$. Отсюда, далее, если обозначим через

$$G(\{x,y\}/\le) \text{ и } \mathcal{D}(\{x,y\}/\le)$$

множества всех $(n-2)$-арных операции X в S, в том же порядке, удовлетворяющих условиям

$$X \in G(\{x,y\}/\le) \overset{\text{деф}}{\Longleftrightarrow} (\forall a_i \in S)_1^{n-2}\, X(a_1^{n-2}) \in$$

$$\in \mathcal{G}(\{x,y\}/\leq_{(a_1^{n-2})}) \text{ и}$$

$$X \in \mathcal{D}(\{x,y\}/\leq) \overset{\text{деф}}{\iff} (\forall a_i \in S)_1^{n-2} X(a_1^{n-2}) \in$$

$$\in \mathcal{D}(\{x,y\}/\leq_{(a_1^{n-2})})$$

для всех $X \in \overrightarrow{S^{n-2}S}$ и каждого $(x,y) \in S^2$, ввиду (γ), находим, что имеют место импликации

$$X \in \mathcal{G}(\{x,y\}/\leq) \Rightarrow (M_{\{x,y\}}(a_1^{n-2}), a_1^{n-2}, X(a_1^{n-2})) \in \leq \quad \text{и}$$

$$X \in \mathcal{D}(\{x,y\}/\leq) \Rightarrow (X(a_1^{n-2}), a_1^{n-2}, m_{\{x,y\}}(a_1^{n-2})) \in \leq$$

для всех $X \in \overrightarrow{S^{n-2}S}$, любого $(a_1^{n-2}) \in S^{n-2}$ и каждого $(x,y) \in S^2$. Таким образом, ввиду определении 2_2 и 2_3, теорема доказана.

ТЕОРЕМА 5. Пусть $(S,\leq)$ R-n-решетка. Пусть, далее,

$$(2) \qquad \cup(x, a_1^{n-2}, y) \overset{\text{деф}}{=\!=} \sup_{\leq}\{x,y\}(a_1^{n-2})[= M_{\{x,y\}}(a_1^{n-2})] \text{ и}$$

$$(3) \qquad \cap(x, a_1^{n-2}, y) \overset{\text{деф}}{=\!=} \inf_{\leq}\{x,y\}(a_1^{n-2})[= m_{\{x,y\}}(a_1^{n-2})].$$

Тогда $(S,\cup,\cap)$ является A-n-решеткой.

ДОКАЗАТЕЛЬСТВО.

Пусть (a_1^{n-2}) любой элемент множества S^{n-2}. Притом, пусть

$$(a) \qquad (x,y) \in \leq_{(a_1^{n-2})} \overset{\text{деф}}{\iff} (x, a_1^{n-2}, y) \in \leq$$

для всех $x,y \in S$. Отсюда, ввиду определения 2_3, находим, что $\leq_{(a_1^{n-2})}$ является РАТ отношением в множестве S для каждого $(a_1^{n-2}) \in S^{n-2}$. Пусть, далее, множества $\mathcal{G}(\{x,y\}/\leq_{(a_1^{n-2})})$ и $\mathcal{D}(\{x,y\}/\leq_{(a_1^{n-2})})$, где x и y любые элементы множества S, определены следующим образом:

$$z \in \mathcal{G}(\{x,y\}/\leq_{(a_1^{n-2})}) \overset{\text{деф}}{\iff} (\exists X \in \overrightarrow{S^{n-2}S})(X \in \mathcal{G}(\{x,y\}/\leq) \wedge$$

$$\wedge\, z = X(a_1^{n-2})) \text{ и}$$

$$z \in \mathcal{D}(\{x,y\}/\leq_{(a_1^{n-2})}) \overset{\text{деф}}{\Longleftrightarrow} (\exists X \in \overrightarrow{S^{n-2}S})(X \in \mathcal{D}(\{x,y\}/\leq) \wedge$$

$$\wedge\ z = X(a_1^{n-2}))$$

для всех $z \in S$ и каждого $(a_1^{n-2}) \in S^{n-2}$; речь идет о множествах всех верхней и нижней грани множества $\{x,y\}$ в $(S,\leq_{(a_1^{n-2})})$. Отсюда, ввиду предположения, что $(S,\leq)$ R-n-решетка и определения (a), находим, что имеют место импликации

$$z \in G(\{x,y\}/\leq_{(a_1^{n-2})}) \Rightarrow (M_{\{x,y\}}(a_1^{n-2}), z) \in \leq_{(a_1^{n-2})} \quad \text{и}$$

$$z \in \mathcal{D}(\{x,y\}/\leq_{(a_1^{n-2})}) \Rightarrow (z, m_{\{x,y\}}(a_1^{n-2})) \in \leq_{(a_1^{n-2})}$$

для всех $z \in S$ и каждого $(a_1^{n-2}) \in S^{n-2}$. Таким образом, если

$$\underparen{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=} M_{\{x,y\}}(a_1^{n-2}) \quad \text{и}$$

$$\overparen{(a_1^{n-2})}(x,y) \overset{\text{деф}}{=} m_{\{x,y\}}(a_1^{n-2})$$

для всех $x,y \in S$, то $(S, \underparen{(a_1^{n-2})}, \overparen{(a_1^{n-2})})$ A-решетка соответствуюущая R-решетке $(S,\leq_{(a_1^{n-2})})$ для каждого $(a_1^{n-2}) \in S^{n-2}$. Отсюда, наконец, учитывая (2) и (3), находим, что теорема доказана.

На табл. 1_1-1_4, 2_1-2_4 описана A-3-решетка $(\{1,2,3,4\}, \cup, \cap)$. Притом: $\cup(x,a,y) \overset{\text{деф}}{=} \underparen{a}(x,y)$ (табл. 1_1-1_4) и $\cap(x,a,y) \overset{\text{деф}}{=} \overparen{a}(x,y)$ (табл. 2_1-2_4). На системе диаграмм 3_1-3_4 изображена соответствующая R-3-решетка $(\{1,2,3,4\},\leq)$. Притом: $(x,a,y) \in \leq \overset{\text{деф}}{\Longleftrightarrow} (x,y) \in \leq_a$.

$\underparen{1}$	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

Табл. 1_1

$\underparen{2}$	1	2	3	4
1	1	2	1	4
2	2	2	2	2
3	1	2	3	4
4	4	2	4	4

Табл. 1_2

$\underparen{3}$	1	2	3	4
1	1	1	3	3
2	1	2	3	4
3	3	3	3	3
4	3	4	3	4

Табл. 1_3

$\underparen{4}$	1	2	3	4
1	1	1	1	1
2	1	2	3	2
3	1	3	3	3
4	1	2	3	4

Табл. 1_4

①	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

Табл. 2_1

②	1	2	3	4
1	1	1	3	1
2	1	2	3	4
3	3	3	3	3
4	1	4	3	4

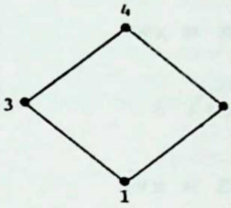
Табл. 2_2

③	1	2	3	4
1	1	2	1	2
2	2	2	2	2
3	1	2	3	4
4	2	2	4	4

Табл. 2_3

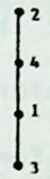
④	1	2	3	4
1	1	2	3	4
2	2	2	2	4
3	3	2	3	4
4	4	4	4	4

Табл. 2_4



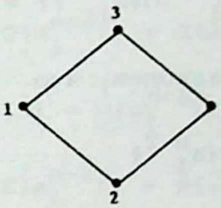
$\leq_1$

Диаг. 3_1



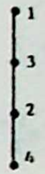
$\leq_2$

Диаг. 3_2



$\leq_3$

Диаг. 3_3



$\leq_4$

Диаг. 3_4

* * *

В [1] автором определенно понятие $\{i,j\}$-нейтральной операции n-группоида; $n \in N \setminus \{1\}$, $i \neq j$, $(i,j) \in \{1,\dots,n\}^2$. В частности, $e: Q^{n-2} \to Q$ является $\{1,n\}$ - нейтральной операцией n-группоида (Q,A) тогда и только тогда, когда имеет место формула:

$$(\varepsilon) \qquad (\forall a_i \in Q)_1^{n-2} (\forall x \in Q)(A(x, a_1^{n-2}, e(a_1^{n-2})) = x \wedge$$

$$\wedge A(e(a_1^{n-2}), a_1^{n-2}, x) = x).$$

Если $n = 2$, то речь идет о нейтральном элементе группоида. Иначе, в n-группоиде существует не больше чем одна $\{1,n\}$-нейтральная операция [1]. Подобно определим $\{1,n\}$-нулевую операцию n-группоида; $n \in N \setminus \{1\}$:

Пусть (Q,A) n-группоид, $n \in N \setminus \{1\}$. $(n-2)$-арную операцию O в множестве Q назовем $\{1,n\}$-нулевой операцией n-группоида (Q,A) тогда и только тогда, когда имеет место формула:

$$(\omega)\qquad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(x,a_1^{n-2},O(a_1^{n-2})) = O(a_1^{n-2})\ \wedge$$
$$\wedge\ A(O(a_1^{n-2}),a_1^{n-2},x) = O(a_1^{n-2})).$$

Если $n = 2$, то речь идет о нулю группоида. Иначе, в n-группоиде существует не больше чем одна $\{1,n\}$-нулевая операция. Именно, из предположения

$$A(\bar{O}(a_1^{n-2}),a_1^{n-2},O(a_1^{n-2})) = O(a_1^{n-2}) \text{ и}$$
$$A(\bar{O}(a_1^{n-2}),a_1^{n-2},O(a_1^{n-2})) = \bar{O}(a_1^{n-2})$$

для любых $a_1^{n-2} \in Q$, находим, что

$$\bar{O}(a_1^{n-2}) = O(a_1^{n-2})$$

для любых $a_1^{n-2} \in Q$.

ТЕОРЕМА 6. Пусть $(S,\cup,\cap)$ A-n-решетка и $|S| \in N$. Тогда существуют (однозначно определение) $(n-2)$-арные операции O и E такие, что

а) O и E $\{1,n\}$-нейтральные операции, в том же порядке, n-группоидов $(S,\cup)$ и $(S,\cap)$; и

б) O и E $\{1,n\}$-нулевые операции, в том же порядке, n-группоидов $(S,\cup)$ и $(S,\cap)$.

ДОКАЗАТЕЛЬСТВО.

Пусть (a_1^{n-2}) любой элемент множества S^{n-2}. Положим:

$$(\alpha)\qquad \undergroup{(a_1^{n-2})}\,(x,y) \overset{\text{деф}}{=\!=} \cup(x,a_1^{n-2},y) \text{ и}$$

$$(\beta)\qquad \overgroup{(a_1^{n-2})}\,(x,y) \overset{\text{деф}}{=\!=} \cap(x,a_1^{n-2},y)$$

для любых $x,y \in S$. Отсюда, учитывая определение 1, находим, что

$$(S, \undergroup{(a_1^{n-2})}, \overgroup{(a_1^{n-2})})$$

решетка для каждого $(a_1^{n-2}) \in S^{n-2}$. Далее, так как $|S| \in N$, находим, что для каждого $(a_1^{n-2}) \in S^{n-2}$ существуют однозначно определение элементы $O(a_1^{n-2})$ и $E(a_1^{n-2})$ удовлетворяющие условиям:

$$(\forall x \in S)(\underbrace{(a_1^{n-2})}(x, O(a_1^{n-2})) = x \wedge$$

$$\wedge \underbrace{(a_1^{n-2})}(O(a_1^{n-2}), x) = x);$$

$$(\forall x \in S)(\overbrace{(a_1^{n-2})}(x, E(a_1^{n-2})) = x \wedge$$

$$\wedge \overbrace{(a_1^{n-2})}(E(a_1^{n-2}), x) = x);$$

$$(\forall x \in S)(\overbrace{(a_1^{n-2})}(x, O(a_1^{n-2})) = O(a_1^{n-2}) \wedge$$

$$\wedge \overbrace{(a_1^{n-2})}(O(a_1^{n-2}), x) = O(a_1^{n-2})); \text{ и}$$

$$(\forall x \in S)(\underbrace{(a_1^{n-2})}(x, E(a_1^{n-2})) = E(a_1^{n-2}) \wedge$$

$$\wedge \underbrace{(a_1^{n-2})}(E(a_1^{n-2}), x) = E(a_1^{n-2}))$$

для каждого $(a_1^{n-2}) \in S^{n-2}$. Отсюда, наконец, учитывая (α) и (β), ввиду (ε) и (ω), находим, что теорема доказана.

А-3-решетка $(\{1,2,3,4\}, \cup, \cap)$ определена на табл. 1_1-1_4, 2_1-2_4 удовлетворяет условию $|S| \in N$ из теоремы 6. Притом опрерации О и Е являются подстановки:

$$O = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \quad \text{и} \quad E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}.$$

ПРИМЕЧАНИЕ

Автор и Б. Шешеля в ряде работ рассматривали различные обобщения РАТ отношении на n-арный случай. Например: On some generalizations of reflexive, antisymetric and tranzitive relations, Proceedings of the Symposium n-ary structures, Skopje 1982, 175-183.

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REZIME

n-MREŽE

U radu se definišu A- i R-n-mreže kao jedno uopštenje A- i R-mreža. Pri tom, u R-n-mrežama tačna gornja i tačna donja ograničenja se definišu kao (n-2)-arne operacije. Utvrđuje se odnos između A- i R-n-mreža i opisuje predstavljanje n-mreža pomoću mreža. Na kraju se dokazuje da u svakoj konačnoj n-mreži postoje (jednoznačno određene) (n-2)-arne operacije O i E takve da su {1,n}-neutralne operacije [1], redom, u n-grupoidima (S,U) i (S,∩), te da su {1,n}-nulte operacije, redom, u n-grupoidima (S,∩) i (S,U).

SUMMARY

n-LATTICES

In this paper we define A- and R-n-lattices as a generalization of A- and R-lattices. In a R-n-lattice the exsact upper and the exact lower bounds are defined as (n-2)-ary operations. A relationship between A- and R-n-lattices is established and a representation of n-lattices using lattices is described. Finally, it is proved that in each finite n-lattice there exist the unique (n-2)-ary operations O and E which are {1,n}-neutral operations [1], respectively, in n-groupoids (S,U) and (S,∩), and also {1,n}-null operations, respectively, in n-groupoids (S,∩) and (S,U).

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О n-КВАЗИГРУППАХ БОЛА

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РЕЗЮМЕ

В работе рассматриваются n-квазигруппы Бола [2], т.е. n-квазигруппы удовливоряющие тождествам под nB_L и nB_R. которые для $n = 2$ превращаются в левое и правое тождество Бола [3 - 5]. В n-квазигруппе Бола (Q,A) существует {1,n}-нейтральная операция [2], т.е. (n-2)-арная операция e удовлетворяющая условию (1); [1]. Таким образом, n-квазигруппы Бола для $n = 2$ являются лупами Муфанг [3-5]. В настоящей работе доказываются некоторые свойства n-квазигрупп Бола и для $n \geq 3$ находится одна характкризация n-квазигрупп Бола.

*

В [1] автором определено понятие {i,j}-нейтральной операции n-группоида; $n \in N \setminus \{1\}$, $i \neq j$, $(i,j) \in \{1,\dots,n\}^2$. В частности, $e : Q^{n-2} \to Q$ является {1,n}-нейтральной операцией n-группоида (Q,A) тогда и только тогда, когда имеет место формула:

$$(1) \qquad (\forall a_i \in Q)_1^{n-2} (\forall x \in Q)(A(x, a_1^{n-2}, e(a_1^{n-2})) = x \land$$

AMS Mathematocs Subject Classification (1980): 20N05.

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$$\wedge\ A(e(a_1^{n-2}),a_1^{n-2},x) = x).^{1)}$$

Если $n = 2$, то речь идет о нейтральном элементе $e(\varnothing)^{1)}$. Иначе, в n-группоиде существует не больше чем одна $\{1,n\}$-нейтральная операция [1].

В [2] автором введены n-квазигруппы Бола. Речь идет о n-квазигруппах$^{2)}$ (Q,A) удовлетворяющих тождествами:

$$nB_L \qquad A(A(\overset{n-1}{x}, A(y,\overset{n-1}{x})),z_1^{n-1}) = A(\overset{n-1}{x},A(y,\overset{n-2}{x},A(x,z_1^{n-1})))$$

и

$$nB_R \qquad A(z_1^{n-1},A(A(\overset{n-1}{x},y),\overset{n-1}{x})) = A(A(A(z_1^{n-1},x),\overset{n-2}{x},y),\overset{n-1}{x}).$$

Если $n = 2$, то в квазигруппе (Q,A) имеют место левое и правое тождество Бола [3-5]. В n-квазигруппе Бола существует $\{1,n\}$-нейтральная операция [2]. Таким образом, n-квазигруппы Бола для $n = 2$ являются лупами Муфанг.

Ввиду теоремы 9 из [2], имеет место следующее утверждение:

ЛЕММА 1. Если (Q,A) n-квазигруппа Бола и $n \geq 3$, то существует группа (Q,B) такая, что

$$A(x,b^{n-2},y) = B(x,B(e(b_1^{n-2})^{-1},y))$$

для любых $x,b^{n-2},y \in Q$, где e $\{1,n\}$-нейтральная операция n-квазигруппы Бола (Q,A) и $^{-1}$ взятие обратного элемента в группе (Q,B).

Притом имеет место и следующее утверждение:

---

[1] Последобательность $g_p,\dots,g_q$ (где $p,\dots,q$ последовательность натуральных чисел следующих друг за другом) для краткости обозначаем через g_p^q; в частности: $g_p^p = g_p$. Подобно, последовательность $\underbrace{a,\dots,a}_{p}$ обозначаем через a. Далее, под g_p^{p-1}, $p \in N$, и под $\overset{\circ}{g}$ понимаем пустую последовательность (отсуствие последовательности). Притом, если речь идет о, например, g_1^3,g_1^0 (или $g_1^3,\overset{\circ}{e}$), то g_1^3,e_1^3 (или $g_1^3,\overset{\circ}{e}$) считаем последовательностю g_1^3, и если речь идет о e_1^0(или $\overset{\circ}{e}$) в некоторых случаях будем обозначать через $\varnothing$; например $F(a_1^0)$ будем обозначать через $F(\varnothing)$.

[2] [6].

ЛЕММА 2. [2]3) Пусть (Q,B) группа, ε отображение множества Q^{n-2} в множество Q и $n \geq 3$. Пусть, далее, для $n=3$ ε подстановка множества Q, и для $n > 3$ (Q,ε) $(n-2)$-квазигруппа. Тогда, если

$$A(x,b_1^{n-2},y) \overset{\text{деф}}{=} B(x,B(\varepsilon(b_1^{n-2})^{-1},y))$$

для любых $x,y,b_1^{n-2} \in Q$, то (Q,A) n-квазигруппа Бола и ε ее $\{1,n\}$-нейтральная операция.

* *

УТВЕРЖДЕНИЕ 3. Пусть (Q,A) n-квазигруппа Бола и $n \geq 3$. Тогда в (Q,A) имеет место тождество $\langle 1,n\rangle$-ассоциативности, т.е. тождество:

$$A(A(x,a_1^{n-2},y),b_1^{n-2},z) = A(x,a_1^{n-2},A(y,b_1^{n-2},z)).$$

Доказательство.

Ввиду леммы 1, находим что имеют место равенства:

$$A(A(x,a_1^{n-2},y),b_1^{n-2},z) =$$

$$= B(B(x,B(e(a_1^{n-2})^{-1},y)),B(e(b_1^{n-2})^{-1},z)) \text{ и}$$

$$A(x,a_1^{n-2},A(y,b_1^{n-2},z)) =$$

$$= B(x,B(e(a_1^{n-2})^{-1},B(y,B(e(b_1^{n-2})^{-1},z))))$$

для любых $x,y,z,a_1^{n-2},b_1^{n-2} \in Q$; e является $\{1,n\}$-нейтральней операцией n-квазигруппы Бола (Q,A).

Отсюда, так как (Q,B) группа, находим, что утверждение доказано.

УТВЕРЖДЕНИЕ 4. Если в n-квазигруппе (Q,A) имеет место тождество $\langle 1,n\rangle$-ассоциативности, то (Q,A) n-квазигруппа Бола.

Доказательство.

Учитывая факт, что в (Q,A) имеет место тождество

---

3) Теорема 10.

<1,n>-ассоциативности, находим, что в (Q,Λ) имеют место следующие равенства

(а) $$\Lambda(\Lambda(a_1^{n-1},\Lambda(a_n^{2n-1})),a_{2n}^{3n-2}) = \Lambda(a_1^{n-1},\Lambda(a_n^{2n-2},\Lambda(a_{2n-1}^{3n-2})))$$

и

(б) $$\Lambda(a_1^{n-1},\Lambda(\Lambda(a_n^{2n-1}),a_{2n}^{3n-2})) = \Lambda(\Lambda(\Lambda(a_1^{n}),a_{n+1}^{2n-1}),a_{2n}^{3n-2})$$

для любых $a_1^{3n-2} \in Q$.

Если, далее, положим

$$a_1^{n-1} = a_{n+1}^{2n-1} = \overset{n-1}{x}, a_n = y, a_{2n}^{3n-2} = z_1^{n-1} \text{ в (а); и}$$

$$a_1^{n-1} = z_1^{n-1}, a_n^{2n-2} = a_{2n}^{3n-2} = \overset{n-1}{x}, a_{2n-1} = y \text{ в (б),}$$

находим, что в (Q,Λ) имеют место тождества под nB_L и nB_R.

Таким образом, учитывая предположение, что (Q,Λ) n-квазигруппа, утверждение доказано.

Следствием утверждения 3 и утверждениа 4 является следующее утверждение:

ТЕОРЕМА 5. Если $n \in N \setminus \{1,2\}$, то n-квазигруппа (Q,Λ) является n-квазигруппой Бола тогда и только тогда, когда в (Q,Λ) имеет место тождество <1,n>-ассоциативности.

Следствием утвержедия 3 является следующее:

УТВЕРЖЕДИЕ 6. Если (Q,Λ) n-квазигруппа Бола и $n \geq 3$, то в (Q,Λ) имеют место следующее тождества:

nLA $$\Lambda(\Lambda(\overset{n}{x}),\overset{n-2}{x},y) = \Lambda(\overset{n-1}{x},\Lambda(\overset{n-1}{x},y));$$

nRA $$\Lambda(\Lambda(x,\overset{n-1}{y}),\overset{n-1}{y}) = \Lambda(x,\overset{n-2}{y},\Lambda(\overset{n}{y})); \text{ и}$$

nE $\Lambda(\Lambda(\overset{n-1}{x},y),\overset{n-1}{x}) = \Lambda(\overset{n-1}{x},\Lambda(y,\overset{n-1}{x}))$. 4)

Учитывая утверждение 6 (nE), находим, что имеет место и следующее утверждение:

УТВЕРЖДЕНИЕ 7. Если (Q,A) n-квазигруппа Бола и $n \geq 3$, то в (Q,A) имеют место следующие тождества

nM1

$$\Lambda(\Lambda(\Lambda(\overset{n-1}{x},y),\overset{n-1}{x}),z_1^{n-1}) =$$

$$= \Lambda(\overset{n-1}{x},\Lambda(y,\overset{n-2}{x},\Lambda(x,z_1^{n-1})));\ \text{и}$$

nM2

$$\Lambda(z_1^{n-1},\Lambda(\overset{n-1}{x},\Lambda(y,\overset{n-1}{x}))) = \Lambda(\Lambda(\Lambda(z_1^{n-1},x),\overset{n-2}{x},y),\overset{n-1}{x}).$$

Для $n = 2$ nM1 и nM2 являются тождествами Муфанг.

Таким образом, утверждение 7 имеет место и для $n = 2$ [3-5].

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REZIME

O n-KVAZIGRUPAMA BOLA

U radu se razmatraju n-kvazigrupe Bola [2], tj. n-kvazigrupe (Q,A),

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4) Утверждение имеет место и для $n = 2$ [3-5]; для $n = 2$ n-квазигруппа Бола является лупой Муфанг.

$n \in N \setminus \{1\}$, u kojima važe zakoni pod nB_L i nB_R, koji se pri $n = 2$ svode na levi i desni Bolov zakon [3-5]. U n-kvazigrupama Bola postoji $\{1,n\}$-neutralna operacija [2], tj. $(n-2)$-arna operacija e koja zadovoljava uslov pod (1). Na taj način, n-kvazigrupe Bola pri $n = 2$ jesu lupe Moufang [3-5]. Svaka n-kvazigrupa Bola pri $n \geq 3$ može se predstaviti superpozicijom jedne binarne grupe i jedne $(n-2)$-kvazigrupe (odnosno permutacije pri $n = 3$)[2]. U ovom radu se utvrđuju neka svojstva n-kvazigrupa Bola i, za $n \geq 3$, nalazi jedna njena karakterizacija.

SUMMARY

ON BOL'S n-QUASIGROUPS

In this paper Bol's n-quasigroups are considered [2], i.e. n-quasigroups (Q,A), $n \in N \setminus \{1\}$, satisfying the laws

$$A(A(\overset{n-1}{x}, A(y, \overset{n-1}{x})), z_1^{n-1}) = A(\overset{n-1}{x}, A(y, \overset{n-2}{x}, A(x, z_1^{n-1})));$$

and

$$A(z_1^{n-1}, A(A(\overset{n-1}{x}, y), \overset{n-1}{x})) = A(A(A(z_1^{n-1}, x), \overset{n-2}{x}, y)^{n-1}),$$

which, for $n = 2$, reduce to the left and right Bol's laws [3-5]. In Bol's n-quasigroups (Q,A) there exists $\{1,n\}$-neutral operation [2], i.e. $(n-2)$-ary operation e satisfying the condition

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(x, a_1^{n-2}, e(a_1^{n-2})) = x \wedge$$
$$\wedge\ A(e(a_1^{n-2}), a_1^{n-2}, x) = x) \qquad [1].$$

In this way, Bol's n-quasigroups for $n = 2$ are Moufang's loops [3-5]. Each Bol's n-quasigroup, $n \geq 3$, can be represented by superposition of one binary group and one $(n-2)$-ary quasigroup (permutation for $n = 3$) [2]. In this paper some properties of Bol's n-quasigroups are established and for $n \geq 3$ a characterization of them is given.

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НЕЙТРАЛЬНЫЕ ОПЕРАЦИИ (n,m)-ГРУППОИДОВ

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РЕЗЮМЕ

В статье вводится понятие $\{i,j\}$-нейтральной операцией в (n,m)-группоиде как одно обобщение $\{i,j\}$-нейтральной операции в n-группоиде [1], и, таким образом, как одно обобщение нульарной операцией - взятие единицы в группоиде (определение 1). Понятие определено для (n,m)-группоидов (Q,A), $A : Q^n \to Q^m$, $(n,m) \in N^2$, удовлетворяющих условию: $n \geq 2m$. Притом: $i \in \{1,\dots,n-2m+1\}$, $j \in \{m+1,\dots,n-m+1\}$ и $j-i \geq m$. В (n,m)-группоиде $(n \geq 2m)$ не существует больше чем одна $\{i,j\}$-нейтральная операция для каждого $(i,j) \in N^2$. Основной результат относится к (n,m)-группам и (n,m)-полугруппам [2-4] (определение 2_1 и 2_2): если $n \geq 3m$, то (n,m)-полугруппа (Q,A) является (n,m)-группой тогда и только тогда, когда (Q,A) обладает $\{1,n-m+1\}$-нейтральной операцией.

Пусть A отображение множества Q^n в множество Q^m и $(n,m) \in (N \cup \{0\}) \times N$. Если $m = 1$, то речь идет о n-арной операцией. Притом, в частности, для $n = 0$ $(m = 1)$ речь идет о нульарной операцией в множестве Q. Иначе, A называется (n,m)-арной операци-

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ей в множестве Q. Если, притом, n = 0, то речь идет о обобщении нульарной операцией $((\Lambda : Q^0 \to Q^m,\ m \in N)$. В примере $(n,m)) \in N^2$, упорядоченная пара (Q,Λ) называется (n,m)-группоидом. Притом, $(n,1)$-группоид является n-группоидом.

Определение 1. Пусть (Q,Λ) (n,m)-группоид и $n \geq 2m$. Пусть, далее, $e_{\{i,j\}}$, где $i \in \{1,\ldots,n-2m+1\}$, $j \in \{m+1,\ldots,n-m+1)$ и $j-i \geq m$, отображение множества Q^{n-2m} в множество Q^m. $(n-2m,m)$-арную операцию $e_{\{i,j\}}$ назовем $\{i,j\}$-нейтральной операцией (n,m)-группоида (Q,Λ) тогда и только тогда, когда имеет место формула:

$$(1)\qquad (\forall a_t \in Q)_1^{n-2m}(\forall x_t \in Q)_1^m(\Lambda(a_1^{i-1},e_{\{i,j\}})a_1^{n-2m}),a_i^{j-m-1},$$

$$x_1^m,a_{j-m}^{n-2m}) = (x_1^m) \wedge$$

$$\wedge\ \Lambda(a_1^{i-1},x_1^m,a_i^{j-m-1},e_{\{i,j\}}(a_1^{n-2m}),a_{j-m}^{n-2m}) = (x_1^m))^{1)}.$$

Если m = 1, то (1) превращается в:

$$(\forall a_t \in Q)_1^{n-2}\ (\forall x \in Q)(\Lambda(a_1^{i-1},e_{\{i,j\}}(a_1^{n-2}),a_i^{j-2},x,a_{j-1}^{n-2})=x$$

$$\wedge\ \Lambda(a_1^{i-1},x,a_i^{j-2},e_{\{i,j\}}(a_1^{n-2}),a_{j-1}^{n-2}) = x).$$

Таким образом, если m = 1, то (Q,Λ) является n-группоидом, $n \in N \setminus \{1\}$, и $e_{\{i,j\}}$ $\{i,j\}$-нейтральной операцией в n-группоиде (Q,Λ), введеной автором в [1].

Если $n = 2m$, то (1) превращается в

$$(\forall x_t \in Q)_1^m(\Lambda(e_{\{1,m+1\}}(\square),x_1^m) = (x_1^m) \wedge$$

$$\wedge\ \Lambda(x_1^m,e_{\{1,m+1\}}(\square)) = (x_1^m))^{1)}.$$

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[1] Последовательность $g_p,\ldots,g_q$ (где $p,\ldots,q$ последовательность натуральных чисел следующих друг за другом) для краткости обозначаем через g_p^q; в частности: $g_p^p = g_p$. Подобно, последовательность $\underbrace{a,\ldots,a}_{p}$ обозначаем через $\overset{p}{a}$. Далее, под g_p^{p-1}, $p \in N$, и под $\overset{0}{g}$ понимаем пустую последовательность (отсуствие последовательности). Притом, если речь идет о, например, g_1^3,e_1^0 (или $g_1^3,\overset{0}{e}$), то g_1^3,e_1^0 (или $g_1^3,\overset{0}{e}$) считаем последовательностю g_1^3, и если речь идет о e_1^0 (или $\overset{0}{e}$) в некоторых случаях будем обозначать через $\square$: например $F(a_1^0)$ будем обозначать через $F(\square)$.

Таким образом, если $n = 2m$, то $i = 1$, $j = m+1$ и $e_{\{1,m+1\}}$ является (o,m)-арной операцией в множестве Q (т.е. $e_{\{1,m+1\}} : Q^o \to Q^m$). Притом, в частности, для $m = 1$ речь идет о нульарной операции $e_{\{1,2\}}$ - взятие единицы $e_{\{1,2\}}(\emptyset)$ в группоиде (Q,A).

Теорема 1. В (n,m)-группоиде, $n \geq 2m$ не существует бильше чем одна $\{i,j\}$-нейтральная операция для каждого $(i,j) \in \in \{(a,b) \mid a \in \{1,\dots,n-2m+1\} \wedge b \in \{m+1,\dots,n-m+1\} \wedge b-a \geq m\}$.

Доказательство.

Пусть существуют $e_{\{i,j\}} : Q^{n-2m} \to Q^m$ и $\bar{e}_{\{i,j\}} : Q^{n-2m} \to Q^m$ удовлетворяющих условиям:

$$A(a_1^{i-1}, e_{\{i,j\}}(a_1^{n-2m}), a_i^{j-m-1}, x_1^m, a_{j-m}^{n-2m}) = (x_1^m) \wedge$$

$$\wedge\, A(a_1^{i-1}, x_1^m, a_i^{j-m-1}, e_{\{i,j\}}(a_1^{n-2m}) a_{j-m}^{n-2m}) = (x_1^m) \quad \text{и}$$

$$A(a_1^{i-1}, \bar{e}_{\{i,j\}}(a_1^{n-2m}), a_i^{j-m-1}, y_1^m, a_{j-m}^{n-2m}) = (y_1^m) \wedge$$

$$\wedge\, A(a_1^{i-1}, y_1^m, a_i^{j-m-1}, \bar{e}_{\{i,j\}}(a_1^{n-2m}), a_{j-m}^{n-2m}) = (y_1^m)$$

для каждого $(a_1^{n-2m}, x_1^m, y_1^m) \in Q^n$.

Отсюда находим, что имеют место равенства

$$A(a_1^{i-1}, e_{\{i,j\}}(a_1^{n-2m}), a_i^{j-m-1}, \bar{e}_{\{i,j\}}(a_1^{n-2m}), a_{j-m}^{n-2m}) =$$

$$= \bar{e}_{\{i,j\}}(a_1^{n-2m}) \quad \text{и}$$

$$A(a_1^{i-1}, e_{\{i,j\}}(a_1^{n-2m}), a_i^{j-m-1}, \bar{e}_{\{i,j\}}(a_1^{n-2m}), a_{j-m}^{n-2m}) =$$

$$= e_{\{i,j\}}(a_1^{n-2m})$$

для любого $(a_1^{n-2m}) \in Q^{n-2m}$, откуда получим, что

$$\bar{e}_{\{i,j\}}(a_1^{n-2m}) = e_{\{i,j\}}(a_1^{n-2m})$$

для каждого $(a_1^{n-2m}) \in Q^{n-2m}$.

Утверждение доказано.

Учитывая факт, что формула под (1) эквивалентна формуле:

$$(\bar{1})\quad (\forall a_t \in Q)_1^{n-2m} (\forall x_t \in Q)_1^m \Lambda(a_1^{i-1}, e_{\{i,j\}}(a_1^{n-2m}), a_i^{j-m-1}, x_1^m, a_{j+m}^{n-2m}) = (x_1^m) \wedge$$

$$\wedge\ (\forall a_t \in Q)_1^{n-2m} (\forall x_t \in Q)_1^m \Lambda(a_1^{i-1}, x_1^m, a_i^{j-m-1}, e_{\{i,j\}}(a_1^{n-2m}), a_{j+m}^{n-2m}) = (x_1^m),$$

подобным способом (из доказательства утверждения 1) доказывается следующее утверждение:

Утверждение 2. Пусть (Q,Λ) (n,m)-группоид, $n \geq 2m$. Пусть, далее, существуют отображения $e_{(i,j)}: Q^{n-2m} \to Q^m$ и $e_{(j,i)}: Q^{n-2m} \to Q^m$, где $i \in \{1,\dots,n-2m+1\}$, $j \in \{m+1,\dots,n-m+1\}$ і $j-i \geq m$, удовлетворяющие условиям:

$$(1_L)\quad (\forall a_t \in Q)_1^{n-2m} (\forall x_t \in Q)_1^m \Lambda(a_1^{i-1}, e_{(i,j)}(a_1^{n-2m}), a_i^{j-m-1}, x_1^m, a_{j+m}^{n-2m}) = (x_1^m) \text{ и}$$

$$(1_R)\quad (\forall a_t \in Q)_1^{n-2m} (\forall x_t \in Q)_1^m \Lambda(a_1^{i-1}, x_1^m, a_i^{j-m-1}, e_{(j,i)}(a_1^{n-2m}), a_{j+m}^{n-2m}) = (x_1^m).$$

Тогда имеют место равенства:

$$e_{(i,j)}(a_1^{n-2m}) = e_{(j,i)}(a_1^{n-2m}) = e_{\{i,j\}}(a_1^{n-2m})$$

для каждого $(a_1^{n-2m}) \in Q^{n-2m}$, где $e_{\{i,j\}}$ $\{i,j\}$-нейтральная операция (n,m)-группоида (Q,Λ).

* *

Обобщая понятие n-группы Дёрнте [5], Г. Чупона в [3] определил понятие (n,m)-группы следующим образом:

Определение 2_1. [3-4][2] Пусть A отображение множества Q^n в множество Q^m, где $n > m$ и $n,m \in N$. Объект (Q,A) называется (n,m)-группой тогда и только тогда, когда (Q,A) удовлетворяет следующим условиям:

$$1° \ (\forall a_i \in Q_i)_1^{2n-m} \ A(A(a_1^n), a_{n+1}^{2n-m}) = A(a_1^j, A(a_{j+1}^{j+n}), a_{j+n+1}^{2n-m})$$

для каждого $j \in \{1,\dots,n-m\}$; и

2° каждое из уравнений $A(x_1^m, a_1^{n-m}) = (b_1^m)$ и $A(a_1^{n-m}, y_1^m) = (b_1^m)$ разрешимо однозначно для любых a_1^{n-m}, $b_1^m \in Q$.

Притом имеет место и следующее опеределение [2]:

Определение 2_2. [2-4] (n,m)-группоид (Q,A), $n > m$, называется (n,m)-полугруппой тогда и только тогда, когда имеет место условие под 1°.

Утверждение 3. Если (Q,A) (n,m)-группа и $n \geq 2m$, то (Q,A) обладает $\{1, n-m+1\}$-нейтральной операцией.

Доказательство.

Пусть (Q,A) (n,m)-группа и $n \geq 2m$. Пусть, далее a_1^{n-2m}, c_1^m любые элементы множества Q. Тогда, ввиду 2° из определения 2_1, уравнение

$$\text{(a)} \qquad A(x_1^m, a_1^{n-2m}, c_1^m) = (c_1^m)$$

обладает единстевенным решением по неизвестной (x_1^m). Решение этого уравнения обозначим через

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[2] В работе под [4] находим обзор результатов по теории (n,m)-групп и (n,m)-полугрупп. Притом, в [4] доказываются новые результаты работающих в Семинаре "Векторско вредносни полугрупи и групи" - Скопje. Иначе, все работы в сборнике "Vector valued semigroups and groups" относятся к теории (n,m)-группоидов.

$$e^{(c_1^m)}_{(1,n-m+1)}(a_1^{n-2m}).$$

Притом, для каждого $(c_1^m) \in Q^m$ $e^{(c_1^m)}_{(1,n-m+1)}$ является отображением множества Q^{n-2m} в множество Q^m.

Пусть, далее, (c_1^m) фиксированный элемент множества Q. Тогда, ввиду 2° из определения 2_1, для каждого $(b_1^m) \in Q^m$ существует в точности один $(y_1^m) \in Q^m$ удовлетворяющий уравнению

$$(б) \qquad (b_1^m) = A(c_1^m, c_1^{n-2m}, y_1^m).$$

Далее, учитывая (а), (б) и 1° из определения 2_1, находим, что имеет место следующая цепь равенств:

$$A(e^{(c_1^m)}_{(1,n-m+1)}(a_1^{n-2m}), a_1^{n-2m}, b_1^m) =$$

$$= A(e^{(c_1^m)}_{(1,n-m+1)}(a_1^{n-2m}), a_1^{n-2m}, A(c_1^m, c_1^{n-2m}, y_1^m)) =$$

$$= A(A(e^{(c_1^m)}_{(1,n-m+1)}(a_1^{n-2m}), a_1^{n-2m}, c_1^m), c_1^{n-2m}, y_1^m) =$$

$$= A(c_1^m, c_1^{n-2m}, y_1^m) = (b_1^m),$$

откуда получаем, что имеет место формула:

$$(\forall a_t \in Q)_1^{n-2m} (\forall b_t \in Q)_1^m \, A(e^{(c_1^m)}_{(1,n-m+1)}(a_1^{n-2m}), a_1^{n-2m}, b_1^m) = (b_1^m).$$

Притом, так как $e^{(c_1^m)}_{(1,n-m+1)}$ одно и то же отображение для каждого $(c_1^m) \in Q^m$, $e^{(c_1^m)}_{(1,n-m+1)}$ обозначим через $e_{(1,n-m+1)}$. В самом деле, мы доказали, что в (n,m)-группе, $n \geq 2m$, существует отображение $e_{(1,n-m+1)} : Q^{n-2m} \to Q^m$ удовлетворяющее условию:

$$(ц) \qquad (\forall a_t \in Q)_1^{n-2m} (\forall x_t \in Q)_1^m \, A(e_{(1,n-m+1)}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = (x_1^m)^{3)}.$$

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[3] см. формулу под (1_L) из утверждения 2 для $i = 1$ и $j = n-m+1$.

Подобным способом доказывается, что существует и отображение $e_{(n-m+1,1)} : Q^{n-2m} \to Q^{m}$, удовлетворяющее условию:

(ц') $(\forall a_t \in Q)_1^{n-2m} (\forall x_t \in Q)_1^{m} A(x_1^m, a_1^{n-2m}, e_{(n-m+1)}(a_1^{n-2m})) =$

$= (x_1^m)$ [4].

Наконец, учитывая факт, что существуют отображения $e_{(1,n-m+1)} : Q^{n-2m} \to Q^m$ и $e_{(n-m+1,1)} : Q^{n-2m} \to Q^m$ удовлетворяющие, в том же порядке, условиям (ц) и (ц'), ввиду утверждения 2, находим, что существует отображение $e_{\{1,n-m+1\}} : Q^{n-2m} \to Q^m$ такое, что

1) $e_{\{1,n-m+1\}} = e_{(1,n-m+1)} = e_{(n-m+1)}$; и

2) $e_{\{1,n-m+1\}}$ является $\{1,n-m+1\}$-нейтральной операцией в (n,m)-группе (Q,A).

Утверждение доказано.

Лемма 4. Пусть (Q,A) (n,m)-полугруппа и (Q,A) обладает $\{1,n-m+1\}$-нейтральной операцией e; $n \geq 2m$ [5]. Тогда, если $n \geq 3m$, то в (Q,A) имеют место импликации:

(α) $A(a_1^m, c_1^{n-m}) = A(b_1^m, c_1^{n-m}) \Rightarrow (a_1^m) = (b_1^m)$; и

(β) $A(c_1^{n-m}, a_1^m) = A(c_1^{n-m}, b_1^m) \Rightarrow (a_1^m) = (b_1^m)$

для любых $a_1^m, b_1^m, c_1^{n-m} \in Q$.

Доказательство.

В каждом (n,m)-группоиде имеет место монотония. Таким образом, в частности, имеют место импликации:

($\bar{\alpha}$) $(a_1^m) = (b_1^m) \Rightarrow A(a_1^m, c_1^{n-m}) = A(b_1^m, c_1^{n-m})$; и

---

[4] см. формулу под (1_R) из утверждения 2 для $i = 1$ и $j = n-m+1$.

[5] определение 1.

$(\bar{\beta})$ $$(a_1^m) = (b_1^m) \Rightarrow \Lambda(c_1^{n-m}, a_1^m) = \Lambda(c_1^{n-m}, b_1^m)$$

для любых $a_1^m, b_1^m, c_1^{n-m} \in Q$.

Пусть a_1^m, b_1^m, c_1^m, d_1^{n-2m} любые элементы множества Q удовлетворяющие равенству:

$$\Lambda(a_1^m, c_1^m, d_1^{n-2m}) = \Lambda(b_1^m, c_1^m, d_1^{n-2m}).$$

Отсюда, ввиду $(\bar{\alpha})$, 1° из определения 2_1 и определения 1, находим, что имеет место следующая цепь импликаций:

$$\Lambda(a_1^m, c_1^m, d_1^{n-2m}) = \Lambda(b_1^m, c_1^m, d_1^{n-2m}) \Rightarrow$$

$$\Lambda(\Lambda(a_1^m, c_1^m, d_1^{n-2m}), e(d_1^{n-2m}), c_1^{n-3m}, e(c_1^m, c_1^{n-3m})) =$$

$$\Lambda(\Lambda(b_1^m, c_1^m, d_1^{n-2m}), e(d_1^{n-2m}), c_1^{n-3m}, e(c_1^m, c_1^{n-3m})) \Rightarrow$$

$$\Lambda(a_1^m, \Lambda(c_1^m, d_1^{n-2m}, e(d_1^{n-2m})), c_1^{n-3m}, e(c_1^m, c_1^{n-3m})) =$$

$$\Lambda(b_1^m, \Lambda(c_1^m, d_1^{n-2m}, e(d_1^{n-2m})), c_1^{n-3m}, e(c_1^m, c_1^{n-3m})) \Rightarrow$$

$$\Lambda(a_1^m, c_1^m, c_1^{n-3m}, e((c_1^m, c_1^{n-3m})) = \Lambda(b_1^m, c_1^m, c_1^{n-3m}, e(c_1^m, c_1^{n-3m})) \Rightarrow$$

$$(a_1^m) = (b_1^m).$$

Таким образом, ввиду факта, что $\vartheta(\perp \Rightarrow p) = \top$, мы доказали, что имеет место импликация

$$\Lambda(a_1^m, c_1^m, d_1^{n-2m}) = \Lambda(b_1^m, c_1^m, d_1^{n-2m}) \Rightarrow (a_1^m) = (b_1^m)$$

для любых $a_1^m, b_1^m, c_1^m, d_1^{n-2m} \in Q$, т.е., что имеет место импликация под (α) для любых $a_1^m, b_1^m, c_1^{n-m} \in Q$.

Подобным способом доказывается, что имеет место импликация под (β) для любых $a_1^m, b_1^m, c_1^{n-m} \in Q$.

Лемма доказана.

Следствием леммы 4 является следующее утверждение:

Лемма 5. Пусть (Q,Λ) (n,m)-полугруппа и (Q,A) обладает

$\{1,n-m+1\}$-нейтральной операцией e; $n \geq 2m$[5]. Тогда, если $n \geq 3m$, то каждое из уравнений

$$A(x_1^m, a_1^{n-m}) = (b_1^m) \quad \text{и} \quad A(a_1^{n-m}, y_1^m) = (b_1^m)$$

(по неизвестных (x_1^m) и (y_1^m)) обладает не больше чем одним решением для любых $a_1^{n-m}, b_1^m \in Q$.

Далее, учитывая лемму 4 и монотонии вида $(\bar{\alpha})$ и $(\bar{\beta})$ из доказательства леммы 4, находим, что имеет место и следующее утверждение:

Лемма 6. Пусть (Q,A) (n,m)-полугруппа и (Q,A) обладает $\{1,n-m+1\}$-нейтральной операцией e; $n \geq 2m$. Тогда, если $n \geq 3m$, то имеют место эквивалентности

$$(\hat{\alpha}) \qquad A(a_1^m, c_1^{n-m}) = A(b_1^m, c_1^{n-m}) \Longleftrightarrow (a_1^m) = (b_1^m) \quad \text{и}$$

$$(\hat{\beta}) \qquad A(c_1^{n-m}, a_1^m) = A(c_1^{n-m}, b_1^m) \Longleftrightarrow (a_1^m) = (b_1^m)$$

для любых $a_1^m, b_1^m, c_1^{n-m} \in Q$.

На основании леммы 6, учитывая доказательство леммы 4, находим, что имеет место следующее утверждение:

Лемма 7. Пусть (Q,A) (n,m)-полугруппа и (Q,A) обладает $\{1,n-m+1\}$-нейтральной операцией e; $n \geq 2m$[5]. Тогда, если $n \geq 3m$, то имеют место эквивалентности

$$A(x_1^m, a_1^m, b_1^{n-2m}) = (c_1^m) \Longleftrightarrow$$

$$(x_1^m) = A(c_1^m, e(b_1^{n-2m}), a_1^{n-3m}, e(a_1^m, a_1^{n-3m})) \quad \text{и}$$

$$A(b_1^{n-2m}, a_1^m, y_1^m) = (c_1^m) \Longleftrightarrow$$

$$(y_1^m) = A(e(a_1^{n-3m}, a_1^m), a_1^{n-3m}, e(b_1^{n-2m}), c_1^m)$$

для любых $a_1^m, b_1^{n-2m}, c_1^m, x_1^m, y_1^m \in Q$.

Ввиду леммы 5, леммы 7 и определения 2_1, находим, что имеет место следующее утверждение:

Утверждение 8. Пусть (Q,A) (n,m)-полугруппа и (Q,A) обладает {1,n-m+1}-нейтральной операцией e; $n \geq 2m$. Тогда, если $n \geq 3m$, то (Q,A) является (n,m)-группой.

Учитывая доказательство леммы 4, ввиду леммы 7 и утверждения 8, находим, что имеет место следующее утверждение:

Утверждение 9. Пусть (Q,A) (n,m)-полугруппа и (Q,A) обладает {1,n-m+1}-нейтральной операцией e; $n \geq 2m$[5]. Тогда, если $n > 3m$, то имеют место равенства

$$A(c_1^m, e(b_1^{n-2m}), \overset{n-3m}{x}, e(a_1^m, \overset{n-3m}{x})) =$$

$$A(c_1^m, e(b_1^{n-2m}), \overset{n-3m}{y}, e(a_1^m, \overset{n-3m}{y})) \quad \text{и}$$

$$A(e(\overset{n-3m}{x}, a_1^m), \overset{n-3m}{x}, e(b_1^{n-2m}), c_1^m) =$$

$$A(e(\overset{n-3m}{y}, a_1^m), \overset{n-3m}{y}, e(b_1^{n-2m}), c_1^m)$$

для любых $a_1^m, b_1^{n-2m}, c_1^m, x, y \in Q$.[6]

Наконец, учитывая утверждение 3 и утверждение 8, находим, что имеет место следующее утверждение:

Теорема 10. Если $n \geq 3m$, то (n,m)-полугруппа (Q,A) является (n,m)-группой тогда и только тогда, когда (Q,A) обладает {1,n-m+1}-нейтральной операцией.

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[6] Для $n = 3m$ $\overset{n-3m}{x}$ и $\overset{n-3m}{y}$ являются пустимы последовательностями.

* * *

Примечание.

Г. Чупона мне обратил внимание на факт, что результат из теормы 10 для $\dot{m} = 1$ (теорема 7 из [1]) в иной форме уже известен. Именно, в [6] и [7], в том же порядке, доказаны следующие утверждения:

а) n-полугруппа $Q([\ ])$ является n-группой $(n \geq 3)$ тогда и только тогда, когда существует одна $(n-2)$-арная операция $(\ )^{-1}$ в Q такая, что для каждых $x_1, \dots, x_{n-2}, y \in Q$ имеют место равенства:

$$[yx_1 \dots x_{n-2}(x_1 \dots x_{n-2})^{-1}] = y =$$

$$= [(x_1 \dots x_{n-2})^{-1} x_1 \dots x_{n-2} y]$$

([6], стр. 11); и

б) Пусть $(G; f, g)$ универсальная алгебра, где f n-арная ассоциативная операция, g $(n-2)$-арная операция и $n > 2$. Тогда имеет место: $(G; f, g)$ является n-группой тогда и только тогда, когда

$$f(y, x_j^{n-2}, g(x_1^{n-2}), x_1^{j-1}) = y \text{ и}$$

$$f(x_k^{n-2}, g(x_1^{n-2}), x_1^{k-1}, y) = y$$

для некоторых $1 \leq k, j \leq n-2$ и всех $y, x_1, \dots, x_{n-2} \in G$ ([7], Corollary 5).

Притом, в [6] и [7] не встречаются сознания о том, что операции $(\ )^{-1}$ и g являются обобщением нульарной операцией - взятие единицы в группоиде, и, что речь идет о однозначно определеных операциях.

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REZIME

NEUTRALNE OPERACIJE (n,m)-GRUPOIDA

U radu se uvodi pojam $\{i,j\}$ -neutralne operacije (n,m)-grupoida kao jedno uopštenje $\{i,j\}$ -neutralne operacije n-grupoida [1], i, na taj način, kao jedno uopštenje nularne operacije - fiksiranje neutralnog elementa u grupoidu (definicija 1). Pojam je definisan za (n,m)-grupoide (Q,A) ($A:Q^n \to Q^m$, $(n,m) \in N^2$) koji zadovoljavaju uslov: $n \geq 2m$. Pri tom: $i \in \{1,\dots,n-2m+1\}$, $j \in \{m+1,\dots,n-m+1\}$ i $j-i \geq m$. U (n,m)-grupoidu za svako $(i,j) \in N^2$. Osnovni rezultat rada odnosi se na (n,m)-grupe i (n,m)-polugrupe [3-4] (definicije 2_1 i 2_2): ako je $n \geq 3m$, onda je (n,m)-polugrupa (Q,A) (n,m)-grupa tada i samo tada, kada (Q,A) ima $\{1,n-m+1\}$-neutralnu operaciju.

SUMMARY

NEUTRAL OPERATIONS ON (n,m)-GROUPOIDS

In this paper the notion of $\{i,j\}$-neutral operation of (n,m)-groupoid as a generalization of $\{i,j\}$-neutral operation of n-groupoid [1] is given. In this way, as a generalization of a nular operation - a neutral element is fixed in the groupoid (Definition 1). This notion is defined for (n,m)-groupoids (Q,A) $(A:Q^n \to Q^m,\ (n,m) \in N^2)$ which satisfy the condition: $n \geq 2m$. In addition: $i \in \{1,\dots,n-2m+1\}$, $j \in \{m+1,\dots,n-m+1\}$ and $j-i \geq m$. In (n,m)-groupoid there is at most one $\{i,j\}$-neutral operation for each $(i,j) \in N^2$. The main result of this paper is concerning (n,m)-groups and (n,m)-semigroups [3-4] (Definitions 2_1 and 2_2): If $n \geq 3m$, then an (n,m)-semigroup (Q,A) is an (n,m)-group if and only if (Q,A) has an $\{1,n-m+1\}$-neutral operation.

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A QUADRATIC SPLINE DIFFERENCE SCHEME FOR A SELF-ADJOINT BOUNDARY VALUE PROBLEM

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Abstract

The exponentially fitted quadratic spline difference scheme for the problem: $-\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $y(0) = \alpha_0$, $y(1) = \alpha_1$ is derived. The scheme has a second order accuracy, under some conditions, on the functions q and f. The numerical results are also given.

1. Introduction

The quadratic spline collocation method ([3], [2]) when applied with a uniform mesh of size h to problem

$$(1)\qquad \begin{cases} Ly = -\varepsilon y'' + q(x)y = f(x), & 0 < x < 1, \\ y(0) = \alpha_0, \quad y(1) = \alpha_1 \end{cases}$$

has the condition

$$(2)\qquad \begin{aligned} & h^2 q_{i+1/2}/\varepsilon \le 1, \quad q_{i+1/2} = q(x_i + \tfrac{h}{2}), \\ & h = x_i - x_{i-1}, \quad i = 1(1)n+1, \quad x_0 = 0, \quad x_{n+1} = 1 \end{aligned}$$

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Key words: quadratic spline, difference scheme, fitting factor, singular perturbation problem.

which leads to spurious oscillation or gross inaccuracies in the approximate solution. In order to avoid this difficulty, we introduce an exponentially fitting factor

$$(3) \qquad \sigma_i = q_{i+1/2} h^2 \left[1 + \frac{2}{sh^2 \rho_i}\right]/8 , \quad \rho_i = h\sqrt{q_{i+1/2}/\varepsilon} ,$$

affecting the second derivative. Because of this the spline collocation method becomes uniformly stable in ε and the corresponding difference scheme is uniformly convergent. The exponentially fitted cubic spline difference scheme for the same problem is derived in [4]. According to [3] the quadratic spline has some adventages over the cubic spline in solving the above problem. The numerical reulsts illustrate this fact.

2. Derivation of the scheme

We shall seek the approximate solution of problem (1) in the form of the quadratic spline $v(x) \in C^1[0,1]$ and on each interval $[x_i, x_{i+1}]$ $v(x)$ has the form

$$v_i(x) = v_i^{(0)} + (x-x_i)v_i^{(1)} + \frac{(x-x_i)^2}{2} v_i^{(2)} .$$

The constants $v_i^{(k)}$, $k=0(1)2$ are obtained from the equations:

$$(4) \qquad \sigma_i v_i^{(2)} + q_{i+1/2}\left[v_i^{(0)} + (x-x_i)v_i^{(1)} + (x-x_i)\frac{v_i^{(2)}}{2}\right]_{x=x_i+\frac{h}{2}} = f_{i+1/2}$$

$$(5) \qquad v_i^{(0)} = v_{i-1}^{(0)} + hv_{i-1}^{(1)} + \frac{h^2}{2} v_{i-1}^{(2)}$$

$$(6) \qquad v_i^{(1)} = v_{i-1}^{(1)} + hv_{i-1}^{(2)}$$

$$v_0 = \alpha_0 , \quad v_{n+1} = \alpha_1$$

From (6) we have $v_{i-1}^{(2)} = \dfrac{v_i^{(1)} - v_{i-1}^{(1)}}{h}$ and, then, from (4) and (5) we have

$$(7) \qquad p_i v_{i+1}^{(1)} + s_i v_i^{(1)} = f_{i+1/2} - q_{i+1/2} v_i$$

$$p_i = -\sigma_i + h^2 q_{i+1/2}/8 , \quad s_i = \sigma_i + 3h^2 q_{i+1/2}/8 .$$

(8) $$v_{i+1}^{(1)} + v_i^{(1)} = 2(v_{i+1} - v_i)/h\ ,\quad v_i = v_i^{(0)}.$$

From (7) and (8) we get

(9) $$v_i^{(1)} = \frac{1}{s_i - p_i}\,[(v_{i+1} - v_i)\cdot 2p_i/h + f_{i+1/2}h - q_{i+1/2}hv_i],\quad i=1(1)n,$$

(10) $$v_{i+1}^{(1)} = \frac{1}{p_i - s_i}\,[f_{i+1/2}h - q_{i+1/2}hv_i^{(0)} + 2s_i(v_{i+1} - v_i)/h],\quad i=0(1)n-1.$$

If we replace the subscript i by $i-1$ in (10) and, then, equalize (9) and (10), we obtain the scheme:

(11) $R_h v_i = Q_h f_i\ ,\quad i=1(1)n$ where

$$R_h v_i = r^- v_{i-1} + r^c v_i + r^+ v_{i+1}$$

$$Q_h f_i = q^- f_{i-1/2} + q^+ f_{i+1/2}$$

$$r^- = -\frac{1}{h(1 + sh^2\rho_{i-1})}\ ,\qquad r^+ = -\frac{1}{h(1 + sh^2\rho_i)}$$

$$r^c = -r^- - r^+ + a_i + a_{i-1},\qquad a_i = \frac{2sh^2\rho_i}{h(sh^2\rho_i + 1)}$$

$$q^+ = q^+(q_i) = \frac{2sh^2\rho_i}{hq_{i+1/2}(1+sh^2\rho_i)}\ ,\qquad q^- = q^+(q_{i-1})$$

3. The truncation error

The truncation error of scheme (11) for an arbitrary smooth function $r(x)$ is defined by

$$\tau_i(r) = R_h r_i - Q_h(Lr)_i$$

Lemma 1. *The truncation error of scheme* (11) *for* $y(x) \in C^3[0,1]$ *has the form:*

$$\tau_i(y) = -(-s_{i-1}\varphi_{i-1,i} + \varphi_{i-1,0})\frac{h}{p_{i-1} - s_{i-1}} +$$

$$+\ \frac{h}{s_i - p_i}(-\varphi_{i,1}p_i + \varphi_{i,0})\ ,\quad \textit{where}$$

$$\varphi_{i,1} = \psi_{i,1} - 2\psi_{i,0}/h\ , \quad \psi_{i,k} = \frac{(h/2)^{3-k}}{(3-k)!}\, y'''(\xi_k), \quad x_{i-1} \le \xi_k \le x_i\ ,$$

$$\varphi_{i,0} = (-\sigma_i + q_{i+1/2}\frac{h^2}{8})\,\psi_{i,1} - q_{i+1/2}\,\psi_{i,0}h + + h\,[(\varepsilon-\sigma_i)y''_{i+1/2} + \sigma_i\psi_{i,2}].$$

Proof.

$\tau_i(y) = R_h(y_i - v_i) = R_h z_i$. The function $z(x)$,

$z(x) = y(x) - v(x) \in C^1[0,1]$, satisfies the following system equations

$$-\sigma_i z_i^{(2)} + q_{i+1/2}\ (z_i^{(0)} + \frac{h}{2}z_i^{(1)} + \frac{h^2}{8} + \psi_{i,0}) = f_{i+1/2} - (\sigma_i - \varepsilon)y''_i$$

$$z_i^{(0)} = z_{i-1}^{(0)} + hz_{i-1}^{(1)} + \frac{h^2}{2}z_{i-1}^{(2)} + \psi_{i,0}$$

$$z_i^{(1)} = z_{i-1}^{(1)} + hz_{i-1}^{(2)} + \psi_{i,0}$$

By eliminating the constants $v_i^{(1)}$ and $v_i^{(2)}$ from the above equations in the same manner as in the derivation of the scheme, we obtain the statement of Lemma 1.

Lemma 2. *The truncation error of scheme* (11) *for the function* $y(x) \in C^2[0,1]$ has the form:

$$\tau_i(y) = \frac{h^2}{2}\, r^- y''(\xi_1) + \frac{h^2}{2}\, r^+ y''(\xi_2) + \varepsilon(q^- y''_{i-1/2} + q^+ y''_{i+1/2}).$$

$$x_{i-1} \le \xi_1 \le x_i \le \xi_2 \le x_{i+1}.$$

Proof. It follows from Taylor developments at x_i.

4. Proof of the uniform convergence

The following lemma from [1] gives us the property of the exact solution which we shall take into account in the proof.

Lemma 3. *Let* $y(x) \in C^4[0,1]$. *Let* $q'(0) = q'(1) = 0$. *Then, the solution of* (1) *has the form*

$$y(x) = u(x) + w(x) + g(x), \quad \textit{where}$$

$$u(x) = a_0\ \exp(-x\sqrt{q(0)/\varepsilon}\ )$$

$$w(x) = a_1\ \exp(-(1-x)\ \sqrt{q(1)/\varepsilon}\ ),$$

$$|g^{(i)}(x)| \le M(1 + \varepsilon^{(1-i/2)i}), \quad i=0(1)4,$$

a_0, a_1 *are bounded functions of* ε *independent of* x, *and* M *is a constant independent of* ε.

Theorem 1. *Let the conditions of Lemma 1 be fulfilled. Let* v_j, $j=0(1)n+1$ *be the approximation to the solution* $y(x)$ *of* (1) *at the grid points obtained by using* (11).

Then, the estimate:

$|y(x_j) - v_j| \le Mh^2$, $j=0(1)n+1$, *where* M *is a constant independent of* ε *and* h, *holds.*

Proof. We shall estimate separately the truncation error for each function u, w and g (see [4]).

Since, $\tau_i(u) = 0$, $\tau_i(w) = 0$ for $q(x) = q(0) = \text{const}$, we have

$$\tau_i(u) = \tau_i(u) - \bar{\tau}_i(u), \quad \tau_i(w) = \tau_i(w) - \bar{\tau}_i(w),$$

where $\bar{\tau}_i(u)$, $\bar{\tau}_i(w)$ are the expressions $\tau_i(u)$ and $\tau_i(w)$ with $q(x) = q(0) = \text{const}$.

After some Taylor developments in the corresponding expressions from Lemma 1, we obtain that

$$|\tau_i(u)|,\ |\tau_i(w)| \le Mh^3/\varepsilon \quad \text{when} \quad h^2 \le \varepsilon.$$

For $\tau_i(g)$ we also use the form given by Lemma 1, and from Lemma 3 we have that

$$|\tau_i(g)| \le Mh^3/\varepsilon, \quad h^2 \le \varepsilon.$$

Thus, (Lemma 3) we have

$$|\tau_i(y)| \le Mh^3/\varepsilon \quad \text{when} \quad h^2 \le \varepsilon. \tag{12}$$

Let $\varepsilon \le h^2$. Then we use the truncation errors for u, w and g in the form given in Lemma 2.

Since all the coefficients in scheme (11) are bounded by Mh^{-1}, we obtain directly that

$$|\tau_i(g)| \le Mh.$$

Using the fact that

$$\tau_i(u) = \tau_i(u) - \overline{\tau}(u), \quad \tau_i(w) = \tau_i(w) - \overline{\tau}(u),$$

$|q(x_i) - q(0)| \le Mx_i^2$, from Lemma 2 and Lemma 3 we have that

$$|\tau_i(u)|,\ |\tau_i(w)| \le Mh, \text{ when } \varepsilon \le h^2.$$

Thus,

$$|\tau_i(y)| \le Mh, \quad \text{for } \varepsilon \le h^2. \tag{13}$$

Denote by Λ the matrix of system (11). Since,

$$|y(x_i) - v_i| \le \|\Lambda^{-1}\| \max_i |\tau_i(y)| \quad \text{and}$$

$$\|\Lambda^{-1}\| \le \max_i |r^c + r^- + r^+|^{-1} \le \begin{cases} M\varepsilon/h, & h^2 \le \varepsilon. \\ Mh, & \varepsilon \le h^2, \end{cases}$$

from (12) and (13) we have that Theorem 1 is valid.

5. Numerical results

Our test for the order of uniform convergence, notation and example are taken from [1]

Example:

$$-\varepsilon y'' + y = -\cos\pi x - 2\varepsilon\pi^2\cos 2\pi x,$$

$$y(0) = 0, \quad y(1) = 1$$

Exact solution:

$$y(x) = \exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})/(1+\exp(-1/\sqrt{\varepsilon})) - \cos^2\pi x$$

Table 1 contains the estimate for the order of uniform convergence.

Table 1

ε \ k	0	1	2	3	4	$\bar{P}\varepsilon$
1/2	2.02	2.00	2.00	2.00	2.00	2.00
1/4	2.02	2.00	2.00	2.00	2.00	2.00
1/8	2.02	2.01	2.00	2.00	2.00	2.01
1/16	2.03	2.01	2.00	2.00	2.00	2.01
1/32	2.03	2.01	2.00	2.00	2.00	2.01
1/64	2.05	2.01	2.00	2.00	2.00	2.01
1/128	2.08	2.02	2.01	2.00	2.00	2.02
1/256	2.14	2.04	2.01	2.00	2.00	2.04
1/512	2.23	2.07	2.02	2.01	2.00	2.06

The computed order of uniform convergence is 2.06 and the classical one is 2.00.

Table 2 contains the maximal differences between the exact and approximate solution at the points of the grid.

Table 3 contains the same results for the cubic spline difference scheme, which is given in [4].

Table 2

$$\max_i |y(x_i) - v_i|$$

ε \ n	16	32	64	128
1/2	.58E-2	.14E-2	.36E-3	.90E-4
1/4	.54E-2	.13E-2	.33E-3	.82E-4
1/8	.48E-2	.12E-2	.29E-3	.73E-4
1/16	.41E-2	.10E-2	.25E-3	.63E-4
1/32	.37E-2	.90E-3	.22E-3	.56E-4
1/64	.34E-2	.84E-3	.21E-3	.52E-4
1/128	.34E-2	.82E-3	.20E-3	.50E-4
1/256	.36E-2	.83E-3	.20E-3	.50E-4
1/512	.38E-2	.84E-3	.20E-3	.50E-4

256	512	1024
.22E-4	.56E-5	.14E-5
.20E-4	.51E-5	.13E-5
.18E-4	.45E-5	.11E-5
.16E-4	.39E-5	.91E-6
.14E-4	.35E-5	.87E-6
.13E-4	.32E-5	.81E-6
.13E-4	.32E-5	.79E-6
.13E-4	.31E-5	.78E-6
.13E-4	.31E-5	.78E-6

Table 3.

n ε	32	64	128	256	512
1/2	0.29E-2	0.72E-3	0.18E-3	0.45E-4	0.11E-4
1/4	0.26E-2	0.66E-3	0.17E-3	0.41E-4	0.10E-4
1/8	0.23E-2	0.59E-3	0.15E-3	0.37E-4	0.92E-5
1/16	0.20E-2	0.51E-3	0.13E-3	0.32E-4	0.79E-5
1/32	0.18E-2	0.45E-3	0.11E-3	0.28E-4	0.70E-5
1/64	0.18E-2	0.42E-3	0.10E-3	0.26E-4	0.65E-5
1/128	0.16E-2	0.40E-3	0.10E-3	0.20E-4	0.53E-5
1/256	0.16E-2	0.40E-3	0.10E-3	0.25E-4	0.63E-5
1/512	0.16E-2	0.40E-3	0.10E-3	0.25E-4	0.63E-5

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Rezime

SPLAJN DIFERENCNA ŠEMA SA KVADRATNIM SPLAJNOVIMA ZA SAMO-ADJUNGOVANI KONTURNI PROBLEM

Izvedena je eksponencijalno fitovana diferencna šema zasnovana na kvadratnim splajnovima za problem: $-\varepsilon y'' + \alpha(x)y = f(x)$, $0 < x < 1$, $y(0) = \alpha_0$, $y(1) = 1$. Šema ima drugi red tačnosti pod određenim uslovima na funkcije q i f. Dati su numerički rezultati.

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SPLINE DIFFERENCE SCHEME ON A NON-UNIFORM MESH

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Abstract

The uniformly convergent cubic spline difference scheme for the problem $Ly = -\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $q(x) \geq q > 0$, $y(0) = \alpha_0$, $y(1) = \alpha_1$ on a non-uniform mesh is derived. The scheme provides for the location of a larger number of points in the boundary layers, while the order of accuracy and the structure of the matrix remain the same as in the uniform grid.

1. Introduction

In [3] a spline difference scheme was constructed on a uniform grid for the problem:

$$(1)\qquad \begin{gathered} Ly = -\varepsilon y'' + q(x)y = f(x), \; 0 < x < 1, \\ y(0) = \alpha_0, \;\; y(1) = \alpha_1, \;\; q(x) \geq q > 0. \end{gathered}$$

It has been proved that the scheme has a second order of uniform accuracy under the condition $q'(0) = q'(1) = 0$. Schemes of the same order of accuracy on a uniform mesh are given in [1], [2] for this problem. Since the spline difference schemes have the same order of precision and the same matrix structure on the uniform and on the non-uniform grid for a fixed ε, we wanted to use this characteristic for singularly perturbed problems. This would enable the modifying of the distribution of the point to the characteristics of the exact solution. Since the condition for the inverse

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monotone of the matrix in ordinary spline difference schemes ([6]) is $h_i^2 q_j/6\varepsilon$, $i=j-1,j$ in [3] the exponential fitting factor was introduced along with the second derivation and the condition was eliminated for the equidistant grid. The determination of the fitting factor for the non-equidistant grid is more complicated. In [3] the fitting factor is determined in such a way that the trunction error of the difference scheme for the function of the boundary layer should be zero. The determining of special conditions for the grid provides for the second order of accuracy.

Proof will be given here of the uniform convergence sheme [3] along with the numerical results which point out the advantage of the non-equidistant grid. The case $q'(0) \neq q'(1)$ will be considered.

2. Derivation of the scheme

Let us substitute differential equations (1) by the equation:

$$-\sigma(x)y''+q(x)y=f(x), \tag{2}$$

where $\sigma(x)$ is the fitting factor which will be determined subsequently. The approximate solution of equation (2) should be sought in the form of a cubic spline $v(x) \in C^2\,[0,1]$:

$$v(x) = v_j + (x-x_j)v_j^{(1)} + \frac{(x-x_j)^2}{2}v_j^{(2)} + \frac{(x-x_j)^3}{6}v_j^{(3)},$$

$$x \in [x_j, x_{j+1}], \quad 0 = x_0 < x_1 < x_2 \dots x_n < x_{n+1} = 1.$$

The points of collocation are the points of the spline grid. Constants are determined from the system of equations:

$$-\sigma_j v_j^{(2)} + q_j v_j = f_j, \quad j = 0(1)n+1, \tag{3}$$

$$v_j = v_{j-1} + hv_{j-1}^{(1)} + \frac{h^2}{2}v_{j-1}^{(2)} + \frac{h^3}{6}v_{j-1}^{(3)}, \quad j = 1(1)n, \tag{4}$$

$$v_j^{(1)} = v_{j-1}^{(1)} + hv_{j-1}^{(2)} + \frac{h^2}{2}v_{j-1}^{(3)}, \quad j = 1(1)n, \tag{5}$$

$$v_j^{(2)} = v_{j-1}^{(2)} + hv_{j-1}^{(3)}, \quad j = 1(1)n, \tag{6}$$

$$v(0) = \alpha_0, \quad v(1) = \alpha_1 .$$

Equation (3) expresses the weakened condition of collocation. Equations (4), (5) and (6) are obtained from the continuity of function $v(x)$. By the elimination of $v_j^{(1)}$, $v_j^{(2)}$, and $v_j^{(3)}$ from the above equations (see [4]) we get the difference scheme:

(7) $$R_h v_j = Q_h f_j, \quad j = 1(1)n,$$

$$R_h v_j = -r_j^- v_{j-1} + r_j^c v_j - r_j^+ v_{j+1},$$

$$Q_h f_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1},$$

$$r_j^- = \left(1 - \frac{h_{j-1} q_{j-1}}{6\sigma_{j-1}}\right) \frac{1}{h_{j-1}}, \quad r_j^+ = \left(1 - \frac{h_j^2 q_{j+1}}{6\sigma_{j+1}}\right) \frac{1}{h_j},$$

$$r_j^c = \left(1 + \frac{h_j^2 q_j}{3\sigma_j}\right) \frac{1}{h_j} + \left(1 + \frac{h_{j-1}^2 q_j}{3\sigma_j}\right) \frac{1}{h_{j-1}},$$

$$q_j^- = \frac{h_{j-1}}{6\sigma_{j-1}}, \quad q_j^c = \frac{h_{j-1} + h_j}{3\sigma_j}, \quad q_j^+ = \frac{h_j}{6\sigma_{j+1}},$$

$$v_0 = \alpha_0, \quad v_{n+1} = \alpha_1, \quad q_j = q(x_j), \quad f_j = f(x_j).$$

In order to get a suitable fitting factor σ_j, we shall use the following lemma.

Lemma 1 [1] *Let* $y \in C^4[0,1]$. *Let* $q'(0) = q'(1) = 0$. *Then the solution of problem* (1) *has the form:*

$y(x) = u(x) + w(x) + g(x)$, *where*

$u(x) = p_0 \exp(-x\sqrt{q(0)/\varepsilon})$,

$w(x) = p_1 \exp(-(1-x)\sqrt{q(1)/\varepsilon})$,

p_0 *and* p_1 *are bounded functions of* ε *independent of* x *and:*

$$|g^{(j)}| \le M\,(1 + \varepsilon^{1-(1/2)j}), \quad j = 0(1)4.$$

M *is a constant independent of* ε.

The local truncation error of scheme (7) for an arbitrary smooth function φ has the form:

$$\tau_j(\varphi) = R_h \varphi_j - Q_h(L\varphi)_j .$$

Lemma 2. *Let in scheme* (7) σ_{j-1} *be replaced by* σ_j^-, σ_j *by* σ_j^c *and* σ_{j+1} *by* σ_j^+ *where:*

$$\sigma_j^- = \frac{h_{j-1}^2 q_{j-1}}{6} S_j(t),$$

$$\sigma_j^c = \frac{h_j h_{j-1}}{6} q_j S_j(t),$$

$$\sigma_j^+ = \frac{h_j^2 q_{j+1}}{6} S_j(t),$$

$$S_j(t) = 1 + \frac{3(h_j + h_{j-1})}{s_j(t)},$$

$s_j(t) = h_j \exp(h_{j-1} t) - (h_j + h_{j-1}) + h_{j-1} \exp(-h_j t)$, *for* $j = 0(1)i_1$,

$s_j(t) = h_j \exp(-h_{j-1} t) - (h_j + h_{j-1} t) + h_{j-1} (\exp(h_j t)$ *for* $i_2 \le j \le n+1$,

$t = \sqrt{q_j/\varepsilon}$, $i_1 \le i_2$, *then:*

a) $\tau_j(u) = 0$, $j = 0(1)i_1$, *for* $q(x) = q = const.$

b) $\tau_j(w) = 0$, $j = i_1(1)n+1$, $i_1 \le i_2$ *for* $q(x) = q = const.$

c) The matrix of system (7) *is an inverse monotone.*

Proof. a), b) By the direct substitution of σ_j, σ_{j-1} and σ_{j+1} in the expression $\frac{h_j h_{j-1}}{h_j + h_{j-1}} R_h u_j$ we obtain zero. Since $Lu_j = 0$, it follows that $\tau_j(u) = 0$. In the same manner it can proved that $\tau_j(w) = 0$.

c) Since $r^- \ge 0$, $r^+ \ge 0$, $r^c > r^- + r^+$ the proof follows.

Throughout the paper M and δ will denote different constants independent of ε and h_j.

Lemma 3. *Let:*

$$(9) \quad \begin{cases} 0 \le h_j - h_{j-1} \le M_j \dfrac{h_{j-1}}{\varepsilon} \min(M_0 h_{j-1}^2, \varepsilon), & 1 \le j \le i_1, \\ 0 \le h_{j-1} - h_j \le M_j \dfrac{h_j}{\varepsilon} \min(M_0 h_j^2, \varepsilon), & i_2 \le j \le n, \end{cases}$$

$\frac{h_j}{h_{j\pm1}} \le M$, $h_j = h = const.$ $\quad i_1 \le j < i_2$, $M_1 \le M$, $i=0(1)n$.

Then, the estimate:

$\max\left(|\sigma_j^- - \varepsilon|, \ |\sigma_j^c - \varepsilon|, |\sigma_j^+ - \varepsilon| \right) \le Mh_j^2$ *holds.*

Proof. Let $\varepsilon \le M_0 h_j^2$. Then:

$$\left| \frac{h_j + h_{j-1}}{s_j(t)} \right| \le M \exp(-h_{j-1} t),$$

$$|S_j(t)| \le M, \quad \frac{h_j}{h_{j\pm1}} \le M, \quad |\sigma_j^c| \le Mh_j^2,$$

$$|\sigma_j^c - \varepsilon| \le Mh_j^2.$$

Let $M_0 h_j^2 \le \varepsilon$. Then, after some Taylor developments we obtain:

$$\frac{3(h_j + h_{j-1})}{s(q_j)} = \frac{6}{h_j h_{j-1} q_j} + 2\,\frac{h_j - h_{j-1}}{h_j h_{j-1}} \sqrt{\frac{\varepsilon}{q_1}} + O(1),$$

$$|\sigma_j^c - \varepsilon| \le Mh_j^2 + M|h_j - h_{j-1}|\sqrt{\varepsilon} \le Mh_j^2 .$$

Since $\frac{h_j}{h_{j-1}} = 1 + \frac{M}{\varepsilon}\min(M_0 h_{j-1}^2, \varepsilon)$ we have $|\sigma_j^- - \varepsilon| \le Mh_j^2$ and $|\sigma_j^+ - \varepsilon| \le Mh_j^2$.

3. Proof of the uniform convergence

Lemma 4. *Let $y \in C^4[0,1]$. Then, the truncation error of scheme (7) has the form:*

$$\tau_j(y) = R_h(y_j - v_j) = -\frac{\varphi_{2,j+1}}{h_j} + \frac{\varphi_{2,j}}{h_{j-1}} - \varphi_{1,j}, \tag{10}$$

$$\varphi_{1,j} = \psi_{1,j} - \frac{h_{j-1}\psi_{2,j}}{2} + \frac{h_{j-1}}{2}\left(\frac{\eta_{j-1}}{\sigma_{j-1}} + \frac{\eta_j}{\sigma_j} \right),$$

$$\varphi_{2,j} = \psi_{0,j} + h_{j-1}^2 \left(\frac{\eta_{j-1}}{3\sigma_{j-1}} + \frac{\eta_j}{6\sigma_j} - \frac{\psi_{2,j}}{6} \right),$$

$$\dot{\eta}_j = y_j''(\sigma_j - \varepsilon),$$

$$\psi_{k,j} = \frac{h_{j-1}^{4-k}}{(4-k)!}\, y^{iv}(\xi_{k,j}), \quad x_{j-1} \le \xi_{k,j} \le x_j .$$

Proof. See [4].

Lemma 5. *Let* $q'(0) = q'(1) = 0$ *in* (1). *Let* $y(x) \in C^4[0,1]$. *Let in (7)* σ_j, σ_{j-1} σ_{j+1} *be replaced by Lemma 2.*

Then:

$$|\tau_j(y)| \le M \frac{h_j}{\varepsilon} \min(M_0 h_j^2, \varepsilon). \tag{11}$$

Proof. According to Lemma 1 we have:

$$\tau_j(y) = \tau_j(u) + \tau_j(w) + \tau_j(g). \tag{12}$$

We shall estimate separately the truncation error of each function u, w, g. We shall start with $u=u(x)$.

Denote by $\tilde{\tau}_j(u)$ the truncation error $\tau_j(u)$ in the case $q(x) = q_0 = q(0) =$ const. The corresponding expressions in $\tilde{\tau}_j(u)$ we mark by ~. Since $\tilde{\tau}_j(u) = 0$ (Lemma 2), we have:

$$\tau_j(u) = \tau_j(u) - \tilde{\tau}_j(u) = \frac{1}{h_j}(-\varphi_{2,j+1} + \tilde{\varphi}_{2,j+1}) + \frac{1}{h_{j-1}}(\varphi_{2,j} - \tilde{\varphi}_{2,j}) -$$

$$- \varphi_{1,j} + \tilde{\varphi}_{1,j},$$

where every σ_{j-1}, σ_j and σ_{j+1} are replaced by σ_j^-, σ_j^c and σ_j^+ respectively.

Let $M_0 h_j^2 < \varepsilon$, then:

$$\left|\frac{1}{\tilde{\sigma}_j^c} - \frac{1}{\sigma_j^c}\right| \le \frac{1}{\tilde{\sigma}_j^c \sigma_j^c}\Big[\frac{h_j h_{j-1}}{6}(q_j - q_0) + \frac{h_j - h_{j-1}}{12}(\sqrt{q_j} - \sqrt{q_c})\sqrt{\varepsilon} +$$

$$+ 0(h_j^2)(q_j - q_0)\Big] \le M\, \varepsilon^{-2} h_j^2 x_j^2,$$

$$|\varphi_{j,1} - \tilde{\varphi}_{j,1}| \le M \frac{h_{j-1}}{\varepsilon}\frac{h_j^2}{\varepsilon^2} x_j^2 \exp(-x_j\sqrt{q_0/\varepsilon}) \le M\frac{h_j}{\varepsilon} h_j^2 \exp(-\delta x_j \sqrt{\varepsilon}),$$

$$|\varphi_{j,2} - \tilde{\varphi}_{j,2}| \le h_{j-1}^2 \left| \frac{\eta_j^-}{3\sigma_{j-1}} - \frac{\tilde{\eta}_j^-}{3\sigma_{j-1}} + \frac{\eta_j^c}{6\sigma_j} - \frac{\tilde{\eta}_j^c}{6\sigma_j} \right| \le M h_{j-1}^2 \frac{h_j^2}{\varepsilon},$$

$|\tau_j(u)| \le M\,(h_j^2/\varepsilon)$, where

$\eta_j^- = y''_{j-1}(\sigma_j^- - \varepsilon)$, $\eta_j^c = y''_j(\sigma_j^c - \varepsilon)$, $\eta_j^+ = y''_{j+1}(\sigma_j^+ - \varepsilon)$.

In the same way we can conclude that:

$$|\tau_j(w)| \le M \frac{h_{j-1}^3}{\varepsilon}. \tag{13}$$

For $\tau_j(g)$ we have $\eta_j^{\pm c} = g_j''(\sigma_j^{\pm c} - \varepsilon) = O(h_j^2)$ and from (10) and Lemma 1 we can see that:

$|\tau_j(g)| \le M \frac{h_j^3}{\varepsilon}$, for $M_0 h_j^2 \le \varepsilon$, and (11) holds.

Let $\varepsilon \le M_0 h_j^2$. Then:

$|s_j(t)| \le M h_j \exp(h_j t)$,

$$\left| \frac{1}{s_j(t)} - \frac{1}{\tilde{s}_j(t)} \right| \le \frac{M}{h_j^2 \exp\left(\frac{h_j}{\sqrt{\varepsilon}} (q_0 - q_j) \right)} \left(\frac{h_{j-1}^2 x_j^2}{\sqrt{\varepsilon}} \exp(-\eta_1) + \frac{h_j^2 x_j^2}{\sqrt{\varepsilon}} \exp(\eta_2) \right),$$

where η_1 is a point between $h_{j-1}\sqrt{q_j/\varepsilon}$, $h_{j-1}\sqrt{q_0/\varepsilon}$, η_2 is a point between $h_j\sqrt{q_j/\varepsilon}$ and $h_j\sqrt{q_0/\varepsilon}$.

Furthermore:

$$\left| \frac{1}{\sigma_j^c} - \frac{1}{\tilde{\sigma}_j^c} \right| \le M\, h_j^{-2} x_j^2 + \frac{x_j^2}{\sqrt{\varepsilon}},$$

$$|\varphi_{j,1} - \tilde{\varphi}_{j,1}| \le \varepsilon\, \frac{h_j}{2}\, u''_j \left| \frac{1}{\sigma_j^c} - \frac{1}{\tilde{\sigma}_j^c} \right| + \varepsilon\, \frac{h_{j-1}}{2}\, u''_{j-1} \left| \frac{1}{\sigma_{j-1}^-} - \frac{1}{\tilde{\sigma}_{j-1}^-} \right|,$$

$$|\varphi_{j,1} - \tilde{\varphi}_{j,1}| \le M\sqrt{\varepsilon}, \quad |\varphi_{j,2} - \tilde{\varphi}_{j,2}| \le M\, h_j \sqrt{\varepsilon}.$$

According to Lemma 4, by the given estimates we have:

$$|\tau_j(u)| \le M\sqrt{\varepsilon}. \tag{14}$$

In a similar manner we obtain:

$$|\tau_j(w)| \le M\sqrt{\varepsilon}. \tag{15}$$

For estimating the truncation error corresponding to function g we use the form:

$$T_j(g) = T_0 g_j + T_1 g_j' - r_j^- \frac{h_{j-1}^2}{2} g''(b_1) - r_j^c + r_j^+ \frac{h_j^2}{2} g''(b_2) -$$

$$- q_j^- q_{j-1} \frac{h_{j-1}^2}{2} g''(b_1) - q_j^+ \frac{h_j^2}{2} g''(b_2) q_{j+1} +$$

$$+ \varepsilon \left(q_j^- g_{j-1}'' + q_j^c g_j'' + q_j^+ g_{j+1}''\right), \quad \text{where} \quad x_{j-1} \le b_1 \le x_j \le b_2 \le x_{j+1} .$$

Since $T_0=0$, $T_1=0$, $|r^{\pm c}| \le M h_j^{-1}$, $|q^{\pm c}| \le M h_j^{-1}$, we have that:

$$(16) \qquad |\tau_j(g)| \le M h_j .$$

Then from (14), (15), (16) and Lemma 1 we obtain (11).

Theorem 1. *Let conditions of Lemma 1 be fulfilled. Let* v_j, $j= 0(1)n+1$ *be the approximation to the solution* $y(x)$ *of* (1) *obtained by using* (7) *with* $\sigma_j, \sigma_{j-1}, \sigma_{j+1}$ *determined by Lemma 2. Let points* x_j, $j=0(1)n+1$ *satisfy conditions* (9).

Then the estimate:

$$(17) \qquad |y(x_j) - v_j| \le M h_j^2, \quad j = 0(1)n+1 \quad \text{holds.}$$

Proof. Let A be the matrix of system (7). A is an inverse monotone matrix and the discrete maximum principle holds. Since:

$$- r_j^- + r_j^c - r_j^+ \ge \begin{cases} M h_j/\varepsilon, & M_0 h_j \ge \sqrt{\varepsilon} , \\ M h_j^{-1}, & M_0 h_j \le \sqrt{\varepsilon} , \end{cases}$$

from (11) it follows that:

$R_h(\pm (y(x_j) - v_j) + M h^{-2}) \ge 0, h^- = max(h_j, h_{j-1}) = const.$, and (17) holds.

In the case $q'(0) \ne q'(1)$, we can prove only the first order of the uniform convergence.

Theorem 2. *Let* $y(x) \in C^4 [0,1]$. *Let* v_j, $j=0(1)n+1$ *be the approximation to the solution* $y(x)$ *of* (1) *obtained by using* (7) *with* σ_{j-1}, σ_j, σ_{j+1} *determined by Lemma 2.*

Let points x_j, $j=0(1)n+1$, *satisfy conditions* (9).

Then:

$$|y(x_j) - v_j| \le M h_j, \quad j = 0(1)n+1.$$

Proof. Function $g(x)$ from Lemma 1 (see [5]) satisfies the following inequality:

$$|g^{(j)}(x)| \le M\,(1 + \varepsilon^{1/2(1-j)}), \quad j = 0(1)4.$$

If we repeat the entire proof of Theorem 1 for this estimate of derivations for $g(x)$ and if we use: $|q(x_j) - q(0)| \le Mx_j$, we shall obtain the statement of Theorem 2.

4. Numerical results

Scheme (7) was used to obtain the approximate solution of problem [1]:

$$-\varepsilon y'' + (1 + x(1-x))y = f(x)$$

$$y(0) = y(1) = 0.$$

Its exact solution is:

$$y(x) = 1 - (1-x)\exp(-x/\sqrt{\varepsilon}) - \exp((x-1)/\sqrt{\varepsilon}).$$

Here we shall present the numerical results which suggest that scheme (7) achieves a uniform second-order accuracy. These results also suggest the choice of the grid.

Table 1, 2 and 3 contain the maximal differeces between the exact and approximate solution at the points of the grid, for different ε and n.

Table 1 contains the results on the equidistant grid.

Table 1

$$\max_j |v_j - y(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.34-04	0.84-05	0.21-05	0.55-06	0.16-06
1/4	0.58-04	0.14-04	0.36-05	0.93-06	0.25-06
1/8	0.80-04	0.20-04	0.50-05	0.13-05	0.34-06
1/16	0.86-04	0.21-04	0.54-05	0.14-05	0.36-06
1/32	0.99-04	0.25-04	0.62-05	0.16-05	0.41-06
1/64	0.13-03	0.33-04	0.82-05	0.21-05	0.54-06
1/128	0.19-03	0.46-04	0.12-04	0.29-05	0.75-06
1/256	0.26-03	0.66-04	0.16-04	0.41-05	0.10-05
1/512	0.42-03	0.95-04	0.23-04	0.58-05	0.15-05
1/1024	0.62-03	0.13-03	0.33-04	0.81-05	0.20-05
1/2048	0.88-03	0.21-03	0.47-04	0.11-04	0.29-05
1/16384	0.18-02	0.60-03	0.15-03	0.33-04	0.83-05

The non-equidestant grid is formed in such a way that more points are found in the layers than outside them.

On the interval $[0,c_0]$, c_0 is a constant which is given in advance, the grid is non-equidistant and obtained according the formula:

$$\tilde{h}_j = \tilde{h}_{j-1} + M\,\frac{\tilde{h}_{j-1}}{\varepsilon}\min(\tilde{h}^2_{j-1},\varepsilon) \quad j = 1(1)n_1-1,$$

$$h_j = Q\tilde{h}_j,\quad Q = c_0/\tilde{Q},\quad \tilde{Q} = \sum_{j=1}^{n_1-1}\tilde{h}_j,\quad x_j = x_{j-1} + h_{j-1},\quad j = 1(1)n_1,\quad x_0 = 0,$$

$\tilde{h}_0$, M and c_0 are given.

On the interval $[c_0, 1-c_0]$ the grid is equidistant:

$$h_j = (1-2c_0)/n_2,\quad j = n_1(1)n_1 + n_2-1,$$

$$n+1 = 2n_1 + n_2,\quad x_j = x_{j-1} + h_j,\quad j = n_1+1(1)n_1 + n_2-1.$$

On the interval $[1-c_0, 1]$ the grid is symmetric to the grid on interval $[0,c_0]$. This is a starting mesh. In each next step the invervals are halved.

Table 2.

$c_0 = 0.05$, $\tilde{h}_0 = 0.03$, $\tilde{M} = 110.5$, $n_1/(n+1) = 1/8$

$$\max_j |v_j - y(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.39-04	0.99-05	0.25-05	0.65-06	0.18-06
1/4	0.82-04	0.21-04	0.52-05	0.13-05	0.35-06
1/8	0.13-03	0.32-04	0.79-05	0.20-05	0.52-06
1/16	0.16-03	0.39-04	0.97-05	0.25-05	0.64-06
1/32	0.21-03	0.54-04	0.13-05	0.34-05	0.87-06
1/64	0.29-03	0.75-04	0.19-05	0.48-05	0.12-05
1/128	0.39-03	0.99-04	0.25-04	0.64-05	0.16-05
1/256	0.46-03	0.11-03	0.29-04	0.74-05	0.19-05
1/512	0.47-03	0.11-03	0.28-04	0.70-05	0.12-05
1/1024	0.45-03	0.98-04	0.24-04	0.60-05	0.15-05
1/2048	0.57-03	0.13-03	0.31-04	0.77-05	0.19-05
1/16384	0.12-02	0.37-03	0.10-03	0.24-04	0.62-05

(About 25 % points in the layers)

Tabla 3.

$c_0 = 0.05,\quad \tilde{h}_0 = 0.01,\quad \tilde{M} = 12,\quad n_1/(n+1) = 1/4$

$$\max_j |v_j - y(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.84-04	0.22-04	0.54-04	0.14-05	0.38-06
1/4	0.18-03	0.46-04	0.12-04	0.29-05	0.76-06
1/8	0.28-03	0.71-04	0.18-04	0.45-05	0.11-05
1/16	0.34-03	0.88-04	0.22-04	0.57-05	0.14-05
1/32	0.47-03	0.13-03	0.32-04	0.83-05	0.21-05
1/64	0.63-03	0.18-03	0.47-04	0.12-04	0.31-05
1/128	0.77-03	0.22-03	0.61-04	0.16-04	0.41-05
1/256	0.84-03	0.24-03	0.66-04	0.17-04	0.45-05
1/512	0.86-03	0.22-03	0.58-04	0.15-04	0.39-05
1/1024	0.62-03	0.13-03	0.32-04	0.82-05	0.21-05
1/2048	0.51-03	0.14-03	0.37-04	0.97-05	0.25-05
1/16384	0.63-02	0.26-03	0.88-04	0.27-04	0.75-05

(About 50 % points in the layers)

From this example it can be seen that about 50 % of the points can be located in the boundary layers while the error remains approximately the same as on the equidistat grid. In addition to this, it can be seen that the results on the nonequidistant grid are better for smaller ε.

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Rezime

SPLAJN DIFERENCNA ŠEMA NA NERAVNOMERNOJ MREŽI

Izvedena je uniformno konvergentna diferencna šema zasnovana na kubnim splajnovima na ravnomernoj mreži za problem $-\varepsilon y'' + q(x)y = f(x)$, $0<x<1$, $q(x) \geq q > 0$, $y(0) = \alpha_0$, $y(1) = \alpha_1$. Šema omogućuje smeštanje većeg broja tačaka u granične slojeve, pri čemu red tačnosti i struktura matrice ostaju isti kao na ravnomernoj mreži.

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SUFFICIENT CONDITIONS FOR A UNIFORM CONVERGENCE FAMILY OF QUADRATIC SPLINE DIFFERENCE SCHEMES

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Abstract

In this paper sufficient conditions for a uniform convergence family of spline difference schemes for the problems $\varepsilon y'' + p(x)y' = f(x)$, $y(0) = \alpha$, $y(1) = \beta$, $x \in [0,1]$ are given. Numerical results are presented.

1. Derivation of schemes

We shall consider the singularly perturbed two point boundary value problems

$$(1)\qquad \begin{gathered} Ly(x) = \varepsilon y''(x) + p(x)y'(x) = f(x), \quad 0 < x < 1, \\ y(0) = \alpha, \quad y(1) = \beta \end{gathered}$$

where ε is a small parameter in $(0,1]$, $p(x)$, $f(x)$ are sufficiently smooth functions and $p(x) \geq p > 0$.

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We want to find an approximation to the solution $y(x)$ of (1) in the form of a quadratic spline $v(x) \in C^1[0,1]$ which has the form:

$$v_j(x) = v_j + (x-x_j)v_j^{(1)} + \frac{(x-x_j)^2}{2}v_j^{(2)}, \quad j = 0(1)n,$$

where $x_j = jh$, $j = 0(1)n+1$, $h = 1/(n+1)$ are the points of an equidistant mesh.

Let us define the fitting "comparison" problem associated with (1) by:

$$(2) \qquad \tilde{L}_j\tilde{y}(x) \equiv \tilde{\sigma}_j(x,\varepsilon)\tilde{y}''(x) + \tilde{p}_j(x)\tilde{y}'(x) = \tilde{f}_j(x), \quad x \in I_j, \quad j=0(1)n,$$
$$\tilde{y}(0) = \alpha, \quad \tilde{y}(1) = \beta,$$

where $\tilde{\sigma}(x,\varepsilon)$ $\tilde{p}(x)$, $\tilde{f}(x)$ are piecewisse constant approximations to a fitting factor $\sigma(x,\varepsilon)$, $p(x)$ and $f(x)$, respectively. $\tilde{\sigma}(x,\varepsilon)$ will be determined.

The unknown coefficients $v_j^{(k)}$, $k=1,2$ are determined from the following conditions.

1. $v(x)$ satisfies (2) at the points $x_{j+1/2} \equiv (x_j + x_{j+1})/2$, $j=0(1)n$ and the boundary conditions,

2. $v(x) \in C^1[0,1]$.

From the above conditions the family of difference schemes was derived in [7].

It is well-known that (1) under condition $p(x) \ge p > 0$ has a unique solution ([1], [2], [5]) $y(x)$ which displays a boundary layer at $x=0$ for small ε. For $p(x)$, $f(x) \in C^3[0,1]$ $y(x)$ can be written in the form.

$$y(x) = u(x) + w(x)$$

where

$$(3) \qquad u(x) = \varepsilon y'(0)\cdot\exp(-p(0)x/\varepsilon)/p(0),$$

$$(4) \qquad |w^{(i)}_{(x)}| \le C\,(1 + \varepsilon^{1-i}\exp(-\delta x/\varepsilon)), \quad i=0(1)4,$$

where δ and C denote constants independent of ε.

Fitting factor $\tilde{\sigma}(x,\varepsilon)$ was determined in [7] from the condition that the truncation error for the boundary layer function $u(x)$ (3) is equal to zero for $p(x) = \tilde{p} = \text{const}$. So,

$$\tilde{\sigma}_j(x,\varepsilon) = \frac{h\tilde{p}_j}{2}\tilde{w}_j, \quad x \in I_j, \quad \text{where } \tilde{w}_j = \text{cth}\,(h\tilde{p}_j/(2\varepsilon)).$$

In that way a family of implicit collocation spline difference schemes was obtained:

(5) $$\tilde{r}_j^- v_{j-1} + \tilde{r}_j^c v_j + \tilde{r}_j^+ v_{j+1} = \tilde{q}_j^- \tilde{f}_j^- + \tilde{q}_j^c \tilde{f}_j^+ , \quad j=1(1)n$$

$$v_0 = \alpha, \quad v_{n+1} = \beta$$

where we denote by $\tilde{f}_j^{\pm}(\tilde{p}_j^{\pm})$ an approximation $f(x)(p(x))$, $x \in I_{j\pm1}$ (for every fixed j).

$$\tilde{r}_j^- = (1-1/\tilde{w}_j^-)/h , \quad \tilde{r}_j^+ = (1-1/\tilde{w}_j^+)/h , \quad \tilde{r}_j^c = -\tilde{r}_j^- - \tilde{r}_j^+ ,$$

$$\tilde{q}_j^- = 1/(\tilde{p}_j^- \tilde{w}_j^-) , \quad \tilde{q}_j^c = 1/(\tilde{p}_j^+ \tilde{w}_j^+), \quad \tilde{w}_j^{\pm} = \operatorname{cth} \frac{h\tilde{p}_j^{\pm}}{2\varepsilon} .$$

Scheme (5) has an abbreviated form

(6) $$\tilde{R}v_j = \tilde{Q}\tilde{f}_j ,$$

where

$$\tilde{R}v_j \equiv \tilde{v}_j^- v_{j-1} + \tilde{r}_j^c v_j + \tilde{r}_j^+ v_{j+1} ,$$

$$\tilde{Q}\tilde{f}_j \equiv \tilde{q}_j^- \tilde{f}_j^- + \tilde{q}_j^c \tilde{f}_j^+ .$$

The choice of the approximation to $p(x)$ and $f(x)$ determines the particular scheme.

Choosing that $\tilde{p}_j^- = \tilde{p}_j^+ = p_j$, $\tilde{f}_j^- = \tilde{f}_j^+ = f_j$, scheme (5) becomes the Allen-Southwell-Il'in scheme for which the first order uniform convergence at the nodes has been proved in [3], [5]. The same scheme was obtained in [6] using a cubic spline $v(x) \in C^1[0,1]$.

Choosing

(7) $$\tilde{p}_j^{\pm} = p_{j\pm1/2} , \quad \tilde{f}_j^{\pm} = f_{j\pm1/2}$$

we get the implicit scheme which, for $\varepsilon = \tilde{\sigma}(x,\varepsilon) = 1$, becomes the scheme corresponding to the spline from [4].

The corresponding difference scheme for

(8) $$\tilde{p}_j^{\pm} = (p_{j\pm1} + p_j)/2 , \quad \tilde{f}_j^{\pm} = (f_{j\pm1} + f_j)/2$$

has been analysed in [7]. The first order uniform and the second order classical convergence has been proved. The same order convergence (as the above schemes) has the following scheme, developed from (5) for

(9) $$\tilde{p}_j^{\pm} = p_{j\pm1} , \quad \tilde{f}_j^{\pm} = f_{j\pm1} .$$

Choosing

$$\tilde{p}_j^- = p_{j-1}\ ,\quad \tilde{p}_j^+ = p_j\ ,\quad \tilde{f}_j^- = f_{j-1}\ ,\quad \tilde{f}_j^+ = f_j\ ,$$

or

$$\tilde{p}_j^- = p_j\ ,\quad \tilde{p}_j^+ = p_{j+1}\ ,\quad \tilde{f}_j^- = f_j\ ,\quad \tilde{f}_j^+ = f_{j+1}\ ,$$

the corresponding schemes have the first order uniform and classical convergence.

2. Sufficient conditions for uniform convergence

The sufficient conditions for a uniform convergence family of cubic spline difference schemes for solving (1) are given in [6].

The proof of the uniform convergence is based on the comparison method developed by Kelloyg & Tsan ([5]) and Berger et al. ([1]).

Lemma 1. *Let* $\{v_j\}$ *be a set of values at the grid points* $\{x_j\}$, $j=0(1)n+1$ *satisfying* $v_0 \le 0$, $v_{n+1} \ge 0$ *and* $\tilde{R}v_j \ge 0$, $j=1(1)n$. *Then* $v_j \le 0$ *for* $j=0(1)n+1$. □

Lemma 2. *If* $K_1(h,\varepsilon) \ge 0$ and $K_2(h,\varepsilon) \ge 0$ *are functions that satisfy:*

$$\tilde{R}(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \ge \tilde{R}(\pm z_j) = \pm\tau_j(y)$$

where $z_j = v_j - y_j$, *then*

$$|z_j| \le K_1(h,\varepsilon)|\phi_j| + K_2(h,\varepsilon)|\psi_j| \ \square$$

As in [1] we use two comparison functions $\phi_j = -2+x_j$, $\psi_j = -\exp(-bx_j/\varepsilon) = -(\mu(b))^j$, where $\mu(b) = \exp(-bh/\varepsilon)$, $b > 0$ will be chosen appropriately.

Throughout the paper $\delta, M, M_1 \ldots$ will be used to denote generic constants independent of h and ε.

Lemma 3. *There are constants* M_1 *and* M_2 *such that for* $h \le M_1$, $0 < b < M_2$ *and* $j=1(1)n$ *the following holds:*

$$\tilde{R}\phi_j \ge \begin{cases} Mh/\varepsilon\ , & h \le \varepsilon \\ M\ , & \varepsilon \le h \end{cases}\ ,\qquad \tilde{R}\psi_j \ge \begin{cases} M\mu^j(b)h/\varepsilon^2, & h \le \varepsilon \\ M\mu^j(b)/h\ , & \varepsilon \le h \end{cases}$$

Proof. See [7]. □

Let in (5)

$$(10)\quad |\tilde{f}_j^{-}-f_{j-1}| \le Mh,\quad |\tilde{f}_j^{+}-f_j| \le Mh,\quad |\tilde{p}_j^{-}-p_{j-1}|,\quad |\tilde{p}_j^{+}-p_j| \le Mh,\quad j=1(1)n,$$

The truncation error for scheme (6) is:

$$\tilde{\tau}_j(y) = \tilde{R}(y_j - v_j) = \tilde{R}y_j - \tilde{Q}\tilde{f}_j =$$

$$= \tilde{R}y_j - \tilde{Q}(Ly_j) + N = \tau_j(y) + N$$

where

$$\tau_j(y) = \tilde{R}y_j - \tilde{Q}(Ly_j) = \tilde{R}y_j - \tilde{q}_j^{-}f_{j-1} - \tilde{q}_j^{c}f_j,$$

$$|N| \le Mh^2/\max(h,\varepsilon).$$

The standard Taylor expansion of $\tau_j(y)$ for fixed ε gives:

$$(11)\quad \tau_j(y) = T_1y_j' + T_2y_j'' + \tilde{r}_j^{+}R_2(x_j, x_{j+1}, y) + \varepsilon\tilde{q}_j^{-}R_0(x_{j-1}, x_j, y'') -$$
$$p_{j-1}\tilde{q}_j^{-}R_1(x_{j-1}, x_j, y')\ ,$$

where

$$(12)\quad R_k(a,b,g) = g^{(k+1)}(\xi)\,\frac{(b-a)^{k+1}}{(k+1)!} = \frac{1}{k!}\int_a^b (b-s)^k g^{(k+1)}(s)ds,\quad \xi \in (a,b).$$

Lemma 4. *Let* (10) *hold, then*

$$|\tau_j(y) \le \begin{cases} M(\frac{h^2}{\varepsilon} + \frac{h^2}{\varepsilon}\exp(-\delta x_j/\varepsilon)), & h \le \varepsilon \\ M\ (h + \exp(-\delta x_{j-1}/\varepsilon)), & \varepsilon \le h, \quad j=1(1)n. \end{cases}$$

Proof. Since $\tau_j(y) = \tau_j(u) + \tau_j(w)$ we shall estimate $\tau_j(u)$ and $\tau_j(w)$ separately.

$$(13)\quad |T_1| = |A + (\tilde{p}_j^{+} - p_j)\,\tilde{q}_j^{c} + \tilde{q}_j^{-}(\tilde{p}_j^{-} - p_{j-1})| \le \begin{cases} Mh^2/\varepsilon, & h \le \varepsilon \\ Mh\ , & \varepsilon \le h. \end{cases}$$

where

$$A = h(\tilde{r}_j^{+} - \tilde{r}_j^{-}) - (\tilde{p}_j^{+}\tilde{q}_j^{c} + \tilde{p}_j^{-}\tilde{q}_j^{-}) \equiv 0,\quad j=1(1)n.$$

$$(14)\quad |T_2| = |h^2(\tilde{r}^{+}_{j} + \tilde{r}^{-}_{j})/2 - \varepsilon(\tilde{q}^{-}_{j} + \tilde{q}^{c}_{j}) - h\tilde{p}^{-}_{j-1}\tilde{q}^{-}_{j})| \le \begin{cases} Mh^2/\varepsilon, & h \le \varepsilon \\ Mh\quad, & \varepsilon \le h. \end{cases}$$

For the coefficients the following estimates hold:

$$(15)\quad |\tilde{r}^{+,c,-}_{j}| \le M/h\ , \qquad |\tilde{q}^{-,c}_{j}| \le \begin{cases} Mh/\varepsilon, & h \le \varepsilon \\ M\ , & \varepsilon \le h \end{cases}, \quad j=1(1)n.$$

From (13), (14), (15) and (4) we get

$$(16)\qquad |\tau_j(w)| \le \begin{cases} M\,(\dfrac{h^2}{\varepsilon} + \dfrac{h^2}{\varepsilon^2}\exp\,(-\delta x_j/\varepsilon)), & h \le \varepsilon, \\ M\,(h + \exp(-\delta x_{j-1}/\varepsilon)), & \varepsilon \le h, \quad j=1(1)n. \end{cases}$$

Let us denote by $\tau_{j,0}(u)$ the truncation error for the boundary layer function $u(x)$ (3) for $p(x) = p(0) = \text{const}$. Since $\tau_{j,0}(u) \equiv 0$, we get

$$\begin{aligned} \tau_j(u) = \tau_j(u) - \tau_{j,0}(u) &= (T_1 - T_{1,0})u'_j + (T_2 - T_{2,0})u''_j + \\ &+ (\tilde{r}^{-}_{j} - \tilde{r}^{-}_{j,0})\,R_2(x_j,x_{j-1},u) + (\tilde{r}^{+}_{j} - \tilde{r}^{+}_{j,0})\,R_2(x_j,x_{j+1},u) - \\ &- (\tilde{q}^{-}_{j} - \tilde{q}^{-}_{j,0})\,\varepsilon R_0(x_j,x_{j-1},u'') - (\tilde{q}^{-}_{j} - \tilde{q}^{-}_{j,0})p_{j-1}R_1(x_j,x_{j-1},u') + \\ &+ (p_{j-1} - p(0))\tilde{q}^{-}_{j,0}\,R_1(x_j,x_{j-1},u')\,. \end{aligned}$$

Using the estimate

$$|\tilde{\sigma}_j - \tilde{\sigma}_{j,0}| = \left|\frac{h\tilde{p}_j}{2}\,(\tilde{w}_j - \tilde{w}_{j,0}) + \tilde{w}_{j,0}\left[\frac{h\tilde{p}_j}{2} - \frac{hp(0)}{2}\right]\right| \le \begin{cases} \varepsilon x_j, & h \le \varepsilon \\ hx_j, & \varepsilon \le h \end{cases},$$

after some Taylor expansions, we get

$$|\tau_j(u)| \le \begin{cases} M\,\dfrac{h^2}{\varepsilon}\exp(-\delta x_j/\varepsilon), & h \le \varepsilon, \\ M\ \exp(-\delta x_{j-1}/\varepsilon), & \varepsilon \le h, \quad j=1(1)n. \end{cases}$$

The statements of Lemma 4 follow from (16) and (17). □

Theorem 1. *Let $\{v_j\}$, $j=0(0)n+1$ be the approximation to the exact solution $y(x)$, $j=0(1)n+1$ of (1) obtained by using (5). Let (10) hold and $p(x), f(x) \in C^3\,[0,1]$, $p(x) \ge p > 0$, then*

$$|v_j - y(x_j)| \le Mh, \quad j=0(1)n+1. \tag{18}$$

Proof. Using the estimates from Lemma 3 and Lemma 4 by Lemma 2 we get (18).□

Theorem 2. *Let the conditions of Theorem 1 hold, then*

$$|v_j - y(x_j)| \le M\frac{h^2}{\varepsilon}, \quad \text{for} \quad h \le \varepsilon, \quad j=0(1)n+1, \tag{19}$$

If the approximations to $p(x)$ and $f(x)$ are (7), (8) or (9).

Proof. For scheme (6) where (8) holds, (19) is proved in [7].

Since

$$(p_{j\pm1} + p_j)/2 = p_{j\pm1/2} + O(h^2), \quad \text{and}$$

$$(f_{j\pm1} + f_j)/2 = f_{j\pm1/2} + O(h^2),$$

the proof for schemes (6),(7) follows from the proof for the previous case.

Let (9) hold, then, by expanding $\tau_j(y)$ up to the derivates of the fourth order, we get

$$|\tau_j(y)| \le M\frac{h^3}{\varepsilon^2} + M\frac{h^3}{\varepsilon^3}\exp(\delta x_j/\varepsilon), \quad \text{for } h \le \varepsilon. \tag{20}$$

Using Lemma 3 and estimate (20) we get (19).

3. Numerical examples

We shall consider the following problems.

I $\quad \varepsilon y'' + y' = x, \quad x \in [0,1],$

$y(0) = y(1) = 0,$

to which the solution is

$$y(x) = (\varepsilon-1/2)(1-\exp(-x/\varepsilon))/(1-\exp(-1/\varepsilon)) - \varepsilon x + x^2/2 .$$

and

II $\quad \varepsilon y'' + (1+x^2)y' = -(e^x + x^2), \quad x \in [0,1] .$

$y(0)=-1, \quad y(1)=0.$

For each problem the mesh length $h=1/J$ was successively halved starting wiht $J=16$ and ending wiht $J=1024$. The maximum error at all the mesh points

$$E_\infty = \max_j |y_j - v_j|$$

is listed under E_∞. The numerical rate of convergence is determined as in [2]:

$$\text{rate} \equiv (\ln z_{k,\varepsilon} - \ln z_{k+1,\varepsilon})/\ln 2,$$

where $z_{k,\varepsilon} = \max_j |v_j^{h/2^k} - v_j^{h/2^{k+1}}|$, $k=0(1)4$,

and $v_j^{h/2^k}$ denotes the value of v_j at the mesh point x_j for the mesh length $h/2^k$.

Numerical results for scheme (8) applied to I and II are given in [7].

Tabel 1. Numerical results for (9) applied to I

	ε	1/2		1/4		1/8		1/16		1/32		1/64		1/128		1/156	
K	J	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate	E_∞	rate
0	16	1.5E-4	2.00	5.4E-4	2.00	1.5E-3	1.99	3.9E-3	1.98	8.3E-3	1.92	1.5E-2	1.72	2.1E-2	1.32	2.5E-2	1.00	2.7E-2	.95
1	32	3.8E-5	2.00	1.3E-4	2.00	3.9E-4	2.00	9.8E-4	1.99	2.1E-3	1.97	4.4E-3	1.92	7.9E-3	1.72	1.1E-2	1.34	1.3E-2	1.03
2	64	9.6E-6	2.00	3.3E-5	2.00	1.0E-4	2.00	2.4E-4	2.00	5.5E-4	1.99	1.1E-3	1.98	2.3E-3	1.91	4.0E-3	1.71	5.7E-3	1.34
3	128	2.4E-6	2.00	8.4E-6	2.00	2.5E-5	2.00	6.2E-5	2.00	1.3E-4	2.00	2.9E-4	1.99	6.1E-4	1.98	1.1E-3	1.92	2.0E-3	1.72
4	256	6.0E-7	2.00	2.1E-6	2.00	6.2E-6	2.00	1.5E-5	2.00	3.5E-5	2.00	7.4E-5	2.00	1.5E-4	1.99	3.1E-4	1.98	6.0E-4	1.92
	512	1.5E-7		5.3E-7		1.5E-6		3.8E-6		8.7E-6		1.8E-5		3.8E-5		7.8E-5		1.5E-4	
	1024	3.7E-8		1.3E-7		3.9E-7		9.7E-7		2.1E-6		4.6E-6		9.7E-6		1.9E-5		3.9E-5	

Table 2. Numerical results for (9) applied to II

ε K	1/2 rate	1/4 rate	1/8 rate	1/16 rate	1/32 rate	1/64 rate	1/128 rate	1/256 rate	1/512 rate
0	2.01	2.01	2.01	1.97	1.83	1.61	1.23	1.02	.99
1	2.00	2.00	2.00	1.99	1.97	1.87	1.60	1.24	1.02
2	2.00	2.00	2.00	2.00	1.99	1.97	1.87	1.60	1.24
3	2.00	2.00	2.00	2.00	2.00	1.99	1.96	1.86	1.60
4	2.00	2.00	2.00	2.00	2.00	2.00	1.99	1.96	1.86

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Rezime

DOVOJNI USLOVI ZA UNIFORMNU KONVERGENCIJU FAMILIJE DIFERENCNIH ŠEMA IZVEDENIH POMOĆU KVADRATNOG SPLAJNA

Dati su dovoljni uslovi za uniformnu konvergenciju familije diferencnih šema za problem $\varepsilon y'' + p(x)y' = f(x)$, $y(0) = \alpha$, $y(1) = \beta$, $x \in [0,1]$. Familija je izvedena pomoću kvadratnog splajna. Prikazani su numerički rezultati.

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MESH GENERATION METHODS FOR NUMERICAL SOLUTION OF QUASILINEAR SINGULAR PERTURBATION PROBLEMS

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Abstract

Quasilinear singularly perturbed boundary value problems are solved numerically by using finite - difference schemes on special non-equidistant meshes. The meshes are generated by suitable functions which redistribute equidistant points. Two similar approaches to mesh generation are compared. "Uniform" convergence is proved for two special types of problems. Numerical results illustrate efficiency of the methods.

1. Introduction

We consider the following singularly perturbed boundary value problem:

$$Tu := -\varepsilon u'' + b(u)u' + c(x,u) = 0\,, \quad x \in I = [0,1]\,, \tag{1.1}$$

$$Ru := (u(0),u(1)) = (U_0,U_1)\,, \tag{1.2}$$

where $' = d/dx$ and ε is a small parameter, $\varepsilon \in (0,1]$ (usually $\varepsilon << 1$). We assume:

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(1.3) $b \in C^1(\mathbb{R})\ ,\quad c \in C^1(I \times \mathbb{R})\ ,$

(1.4) $c_u(x,u) \geq c_* > 0\ ,\ x \in I\ ,\ u \in \mathbb{R}\ .$

Then there exist numbers $\bar{u}$, $\underline{u}$, such that $\bar{u} > \underline{u}$ and:

(1.5) $c(x,\underline{u}) \leq 0 \leq c(x,\bar{u})\ ,\ x \in I\ ,$

(1.6) $\underline{u} \leq U_j \leq \bar{u}\ ,\ j=0,1\ ,$

and since the operator (T,R) is inverse monotone, there exists a unique solution u_ε, to the problem (1.1), (1.2), see [10], [11]. Moreover, for $x \in I$ we have

$$u_\varepsilon(x) \in W := [\underline{u},\bar{u}]\ ,$$

and

$$u_\varepsilon \in C^3(I)\ .$$

It is well known (see [4], [14], for instance) that u_ε may have one or more boundary or/and interior layers. Our aim is to solve (1.1), (1.2) numerically by using finite-difference schemes on special non-equidistant meshes. The meshes should be dense in the layers. On one hand this will give us a high percentage of numerical values in the regions where u_ε changes abruptly. On the other hand, we can prove a sort of uniform convergence (i.e. convergence uniform in ε) of the numerical solution towards u_ε, provided some additional information about bahaviour of u_ε be available.

Let us give some further details about our results and let us introduce some notation. First of all, we shall use (1.1) in the conservation form:

(1.7) $Tu = -\varepsilon u'' + f(u)' + c(x,u) = 0$

where

$$f_u(u) = b(u)\ .$$

We shall consider two different, but similar, ways of introducing non-equidistant meshes. We refer to them as *direct* and *indirect* approaches. In the direct approach we discretize (1.7) on some *non-equidistant* mesh I_h with the mesh points:

$$0 = x_0 < x_1 < \ldots < x_n = 1\ ,\quad n \in \mathbb{N}\ .$$

We shall consider meshes generated by suitable functions λ, i.e.:

$$(1.8)\qquad x_i = \lambda(t_i)\ ,\quad t_i = ih\ ,\quad i=0,1,\dots,n\ ,$$

where $h=1/n$ and λ is sufficiently smooth function (at least from $C^1(I)$), which usually depends on ε and satisfies

$$(1.9)\qquad \lambda'(t) > 0\ ,\quad t \in I\ ;\quad \lambda(s)=s\ ,\quad s=0,1\ .$$

In the indirect approach we introduce a new independent variable t by $x=\lambda(t)$ and transform (1.7). Then the transformed problem is discretized on *equidistant* mesh $\tilde{I}_h$ with mesh points t_i. We use the notation (1.8) in this approach as well and assume that λ has the same properties as above.

Let

$$h_i = x_i - x_{i-1}\ ,\quad i=1,2,\dots,n\ ,$$

$$\bar{h}_i = (h_i + h_{i+1})/2\ ,\quad i=1,2,\dots,n-1,$$

and let w_h, v_h, etc. denote arbitrary mesh functions on $I_h\setminus\{0,1\}$ (or $\tilde{I}_h\setminus\{0,1\}$) which will be identified with R^{n-1} - column vectors. Thus,

$$w_h = [w_1, w_2, \dots w_{n-1}]^T\ ,\ (w_i := w_{h,i}\ ,\quad i=1,2,\dots,n-1).$$

By $w_{\varepsilon,h}$ we shall denote the numerical solution with components $w_{\varepsilon,i}$ which approximate $u_\varepsilon(x_i)$. Let

$$u_{\varepsilon,h} = [u_\varepsilon(x_1),\ u_\varepsilon(x_2),\dots,u_\varepsilon(x_{n-1})]^T\ .$$

Let $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote usual vector (matrix) norms in $R^{n-1}(R^{n-1,n-1})$. Define $\|\cdot\|_h$:

$$\|w_h\|_h = \sum_{i=1}^{n-1} \bar{h}_i\,|w_i| = \|Hw_h\|_1\ ,$$

where $H = \operatorname{diag}(\bar{h}_1,\ \bar{h}_2,\dots,\ \bar{h}_{n-1})$. The norm $\|\cdot\|$ is the standard L^1 discrete norm on non-equidestant meshes, cf. [1]. The corresponding matrix norm is:

$$(1.10)\qquad \|A\|_h = \|HAH^{-1}\|_1\ ,$$

where $A \in R^{n-1,n-1}$.

In both approaches we prove

$$\|w_{\varepsilon,h} - u_{\varepsilon,h}\|_h \le M_\varepsilon h , \tag{1.11}$$

where M_ε stands for any positive constant which is independent of h (but may depend on ε). We can prove (1.11) easily since we use stable discrete operators (Engquist - Osher and Lax - Friedrichs schemes, in particular). For instance, in the direct approach we prove the following stability inequality:

$$\|w_h - v_h\|_h \le c_*^{-1} \|T_h w_h - T_h v_h\|_h , \tag{1.12}$$

where T_h is a discrete operator corresponding to T from (1.7), $T_h : R^{n-1} \to R^{n-1}$. A similar result holds in the case of the indirect approach, as well.

The inequality (1.11) shows the linear discrete L^1 convergence of the numerical solution to the exact solution. The convergence is by no means uniform in ε: a constant M_ε is involved and even the norm $\|\cdot\|_h$ depends on ε if h_i's depend on it (which is the case when the mesh is dense in the layers for all values of ε). We can't avoid both causes of the non - uniformity and our aim is to use a special mesh (i.e. a special function λ) to get

$$\|w_{\varepsilon,h} - u_{\varepsilon,h}\|_h \le Mh , \tag{1.13}$$

where by M we denote any positive constant which is independent of h and ε. Although we use $\|\cdot\|_h$ which depends on ε, we shall refer to (1.13) as to the uniform convergence. Of course, it is not easy to obtain (1.13). We have to prove that the norm $\|\cdot\|_h$ of the consistency error of the discrete operator is bounded by Mh. This can be done only if we have sufficiently sharp estimates of the derivatives of u_ε (which occur in the consistency error) and if we use an appropriate function λ which condenses the mesh in the layers. Essentially, any part of λ which maps the mesh points into a layer behaves like inverse function of the corresponding boundary/interior layer function. Then, such parts are smoothly connected by appropriate polynomials.

Let us now introduce some more notation. Let

$$e_h = [1,1,\dots,1]^T \in R^{n-1} ,$$

$$W_h = \{ w_h \in R^{n-1} \mid \underline{u} e_h \le w_h \le \bar{u} e_h \} .$$

The inequality sign in R^{n-1} should be understood componentwise.

Finally, let $g\colon R^2 \to R$, $g \in C^1(R^2)$. The following properties of g will be of interest:

$$g_v(u,v) \ge 0 \ge g_u(u,v)\ , \quad u,v \in R\ , \tag{1.14}$$

$$g(u,u) = f(u)\ , \quad u \in W\ , \tag{1.15}$$

$$|g_u(u_1,v) - g_u(u_2,v)| \le G|u_1 - u_2|\ , \quad u_1,u_2,v \in W\ , \tag{1.16}$$

$$|g_v(u,v_1) - g_v(u,v_2)| \le G|v_1 - v_2|\ , \quad u,v_1,v_2 \in W\ , \tag{1.17}$$

with some $G \ge 0$. The function g will be used for discretization of the transport term in (1.7), e.g. in the direct approach:

$$f(u)'(x_i) \approx \Big[g(u(x_{i+1}),\ u(x_i)) - g(u(x_i),\ u(x_{i-1})) \Big] / \bar{h}_i\ ,$$

cf. [1], [11-13].

The paper consists of 5 sections. After the Introduction, in Section 2 we consider the approach of direct discretization. First we prove the stability (1.12). This is a well known result for the class of schemes which we consider (cf. [1]), but we give different and simpler proof which uses the techniques of M - functions, see [3], [11-13]. Then we consider the consistency error and derive conditions on λ which imply (1.13). However, from these conditions it is not easy to see how to choose λ. The indirect approach, which we consider in Section 3, gives more information about the choice of λ. Section 3 contains the analysis of stability and consistency, similar to Section 2. In Section 4 we consider two special types of problem (1.1), (1.2), (a non-turning point problem and a boundary shock problem), which enables us to prove (1.13). A closer attention is kept on the direct approach, since these problems have already been solved numerically in [23] and [24] by the indirect approach. Finally, in Section 5 we present some numerical results and give some concluding remaks.

The mesh generation methods which we consider here date from 1969, [2]. In that paper the direct approach was introduced. The indirect approach was considered in [8] and [9], for instance. Other papers which use generated meshes are [7], [17-21], [25]. All of these papers deal with different types of linear or semilinear singular perturbation problems.

In general, mesh construction methods can be devided into implicit and explicit methods. In the explicit methods the mesh is given in advance. Thus, the methods which we use here belong to this class. Another explicit method is the method of Ascher and Weiss, see [26] for instance. The implicit methods, on the other hand, give as final numerical result both the numerical solution and the mesh points. To this class belong simultaneous methods, see [27], and alternate methods [6], [16].

2. The approach of direct discretization

Consider the following finite-difference operators on the mesh I_h :

$$D''_h z_i = (h_{i+1} z_{i-1} - 2\bar{h}_i z_i + h_i z_{i+1})/(h_i h_{i+1} \bar{h}_i) \ ,$$

$$D'_h z_i = \Big[g(z_{i+1}, z_i) - g(z_i , \ z_{i-1}) \Big] / \bar{h}_i \ ,$$

where $\{z_i\}$ is a mesh function on I_h and g has been introduced is Section 1. Let T_h, T_h: $R^{n-1} \to R^{n-1}$, be given by:

$$T_h w_i := (T_h w_h)_i = - \varepsilon D''_h w_i + D'_h w_i + c(x_i, \ w_i) \ , \quad i=1,2,\dots,n-1 \ ,$$

where w_0 and w_n should be replaced by U_0 and U_1, respectively. Then the discretization of the problem (1.7), (1.2) reads:

$$T_h w_h = 0 \ . \tag{2.1}$$

Theorem 2.1 *Let* (1.3), *(without* b_u, $c_x \in C(R)$*)*, (1.4) *and* (1.14) *hold. Then there exists a unique solution* $w_{\varepsilon,h}$ *to the problem* (2.1), *and* $w_{\varepsilon,h} \in W_h$. *Moreover, for any* w_h, v_h *the stability inequality* (1.12) *holds.*

Proof. Let $T'_h(w_h)$ denote the Frechet derivative of the operator T_h. The i-th column ($i=1,2,\dots,n-1$) of $T'_h(w_h)$ has the following non-zero elements:

$$T'_h(w_h)_{i-1,i} = -\varepsilon/(h_i \bar{h}_{i-1}) + g_u(w_h, w_{i-1})/\bar{h}_{i-1} \ ,$$

$$T'_h(w_h)_{i,i} = 2\varepsilon/(h_i h_{i+1}) + \Big[g_v(w_{i+1}, w_i) - g_u(w_i, w_{i-1}) \Big] / \bar{h}_i + c_u(x_i, w_i) \ ,$$

$$T'_h(w_h)_{i+1,i} = -\varepsilon/(h_{i+1} \bar{h}_{i+1}) - g_v(w_{i+1}, w_i)/\bar{h}_{i+1} \ ,$$

where we take formally: $T'_h(w_h)_{0,1} = 0$ and $T'_h(w_h)_{n,n-1} = 0$.

Then because of (1.14) $T'_h(w_h)$ is an L - matrix and:

$$(\bar{h}_{i-1}/\bar{h}_i)T'_h(w_h)_{i-1,i} + T'_h(w_h)_{i,i} + (\bar{h}_{i+1}/\bar{h}_i)T'_h(w_h)_{i+1,i} \ge c_u(x_i,w_i)\ ,$$

$$i=1,2,\dots,n-1,$$

hence

$$(HT'_h(w_h)H^{-1})^T e_h \ge c_* e_h\ .$$

Thus $T'_h(w_h)$ is an M-matrix and

$$\|((HT'_h(w_h)H^{-1})^T)^{-1}\|_\infty \le 1/c_*\ ,$$

i.e. (see (1.10)):

$$\|(T'_h(w_h))^{-1}\|_h \le 1/c_*\ .$$

Thus T_h is a homeomorphism, [15], and because of

$$w_h - v_h = \left[\ \int_0^1 T'_h(v_h + s(w_h - v_h))\,ds\right]^{-1}(T_h w_h - T_h v_h)\ ,$$

we get (1.12). Finally, from (1.5), (1.6) we get:

$$T_h(\bar{u}e_h) \ge 0 \ge T_h(\underline{u}e_h)\ ,$$

hence $w_{\varepsilon,h} \in W_h$. □

Now let us consider the consistency error r_i, $i=1,2,\dots,n-1$, of the operator T_h:

$$r_i = (Tu_\varepsilon)(x_i) - T_h u_\varepsilon(x_i)\ .$$

We have

$$r_i = r''_i + r'_i\ ,$$

where

$$r''_i = \varepsilon(D''_h u_\varepsilon(x_i) - u''_\varepsilon(x_i))\ ,$$

$$r'_i = f(u_\varepsilon)'(x_i) - D'_h u_\varepsilon(x_i)\ .$$

We shall use integral forms of r''_i and r'_i, cf. [5]. Let us introduce some notation:

$$J(i,j,k) = \int_{x_i}^{x_j} (s-x_j)^{k-1} u_\varepsilon^{(k)}(s)\,ds\ ,$$

$$g_{u,i} = g_u(u_\varepsilon(x_i), u_\varepsilon(x_i)) \ ,$$

$$g_{v,i} = g_v(u_\varepsilon(x_i), u_\varepsilon(x_i)) \ .$$

The following lemma is easy to prove:

Lemma 2.1 *Let* (1.3) *and* (1.15) *holds. Then for* $i=1,2,\dots,n-1$ *we have:*

$$r_i'' = \varepsilon(2h_i\bar{h}_i)^{-1}J(i,i-1,3) + \varepsilon(2h_{i+1}\bar{h}_i)^{-1}J(i,i+1,3) \ ,$$

$$\begin{aligned} r_i' = {} & g_{u,i}\Big[(h_i - h_{i+1})(2h_{i+1}\bar{h}_i)^{-1}J(i,i+1,1) + h_{i+1}^{-1}J(i,i+1,2)\Big] + \\ & + g_{v,i}\Big[(h_i - h_{i+1})(2h_i\bar{h}_i)^{-1}J(i,i-1,1) + h_i^{-1}J(i,i-1,2)\Big] + \\ & + \bar{h}_i^{-1}\Bigg[\int_{u_\varepsilon(x_i)}^{u_\varepsilon(x_{i+1})}(g_{u,i} - g_u(s,u_\varepsilon(x_i)))ds + \\ & + \int_{u_\varepsilon(x_{i-1})}^{u_\varepsilon(x_i)}(g_{v,i} - g_v(u_\varepsilon(x_i),s))ds\Bigg] \ . \end{aligned}$$

Let us now consider the discretization mesh I_h given by (1.8). Additionally to (1.9) assume that

$$\lambda'(t) \le M \ , \quad t \in I \ , \tag{2.2}$$

and that λ'' be piecewise continuous in I and

$$|\lambda''(t)| \le M \ , \tag{2.3}$$

in any subinterval of I where λ'' is continuous. From (2.2) we have

$$M_\varepsilon h \le h_i = \lambda(t_i) - \lambda(t_{i-1}) \le Mh \ , \quad i=1,2,\dots,n \ , \tag{2.4}$$

and from (2.3):

$$|h_{i+1} - h_i| = |\lambda(t_{i+1}) - 2\lambda(t_i) + \lambda(t_{i-1})| \le Mh^2 , \tag{2.5}$$

$$i=1,2,\dots,n-1 ,$$

cf. [22, Theorem 8].

Theorem 2.2 *Let* (1.3), (1,4) *and* (1.14-17) *hold and let the mesh* I_h *be given by* (1.8), *where* λ *satisfies* (1.9), (2.2), (2.3). *Then the convergence* (1.11) *holds.*

Proof. Since

$$r_i = T_h w_{\varepsilon,i} - T_h u_\varepsilon(x_i) , \quad i=1,2,\dots,n-1 ,$$

the result follows from Theorem 2.1, Lemma 2.1 and (2.4), (2.5). □

Finally, we shall give the theorem on the uniform convergence. The proof is obvious.

Theorem 2.3 *Let the conditions of Theorem* 2.2 *hold and let the following inequalities be satisfied for* $i=1,2,\dots,n-1$:

$$\varepsilon h_i^{-1}|J(i,i-1,3)| , \ \varepsilon h_{i+1}^{-1}|J(i,i+1,3)| \le Mh^2 , \tag{2.6}$$

$$h_i^{-1}|J(i,i-1,1)| , \ h_{i+1}^{-1}|J(i,i+1,1)| \le M , \tag{2.7}$$

$$(\bar{h}_i/h_i)|J(i,i-1,2)| , \ (\bar{h}_i/h_{i+1})|J(i,i+1,2)| \le Mh^2 . \tag{2.8}$$

Then the uniform convergence (1.13) *holds.*

The following well known schemes satisfy the conditions (1.14-17):

- the Lax - Friedrichs (LF) scheme:

$$g(u,v) = (1/2)(f(u) + f(v) + B(v-u)) ,$$

where B is such a number that

$$|b(u)| \le B , \quad u \in W .$$

- the Engquist - Oscher (EO) scheme:

$$g(u,v) = \int_0^u b_-(s)ds + \int_0^v b_+(s)ds ,$$

where

$$b_-(s) = \min\{0,b(s)\} , \quad b_+(s) = \max\{0,b(s)\} .$$

In particular, EO scheme satisfies (1.16), (1.17) since:

$$|b_-(u_1) - b_-(u_2)| \le |b(u_1) - b(u_2)| ,$$

$$|b_-(v_1) - b_-(v_2)| \le |b(v_1) - b(v_2)| ,$$

Other properties can be checked easily.

3. The indirect approach

Let $\lambda \in C^2(I)$ and let (1.8) and (1.9) hold throughout this section. Let t and $\tilde{u}$ be new variables introduced by

$$x = \lambda(t) , \quad \tilde{u}(t) = u(\lambda(t)) .$$

Let $\rho(t) = 1/\lambda'(t)$. Then the problem (1.7), (1.2) is transformed to:

$$\tilde{T}\tilde{u} := -\varepsilon(\rho(t)\tilde{u}')' + f(\tilde{u})' + \lambda'(t)c(\lambda(t),\tilde{u}) = 0 , \quad t \in I , \tag{3.1}$$

$$\tilde{u}(0) = U_0 , \ \tilde{u}(1) = U_1 . \tag{3.2}$$

Here $' = d/dt$. By $\tilde{u}_\varepsilon$ we shall denote the solution to (3.1), (3.2); $\tilde{u}_\varepsilon(t) = u_\varepsilon(\lambda(t))$.

To discretize (3.1), (3.2) we use the equidistant mesh $\tilde{I}_h$. The discretization reads:

$$\tilde{T}_h w_i := - \varepsilon\tilde{D}''_h w_i + \tilde{D}'_h w_i + \lambda'(t_i)c(\lambda(t_i),w_i) = 0 , \tag{3.3}$$

$$i=1,2,\dots,n-1 ,$$

where

$$\tilde{D}''_h w_i = h^{-2}\Big[\rho(t_{i-1/2})(w_{i-1}-w_i) + \rho(t_{i+1/2})(w_{i+1}-w_i)\Big] ,$$

$t_{i+1/2} = t_i \pm h/2$, and $\tilde{D}'_h$ is the same as D'_h from Section 2 in the case of equidistant mesh. As before, w_0 and w_n in (3.3) should be replaced by U_0 and U_1, respectively.

Theorem 3.1 *Let the conditions of Theorem 2.1 hold. Then there exists a unique solution* $w_{\varepsilon,h}$ *to the problem* (3.3) *and* $w_{\varepsilon,h} \in W_h$. *Moreover, for any* w_h, v_h *the following stability inequality holds:*

$$\sum_{i=1}^{n-1} \lambda'(t_i)\,|w_i - v_i| \le c_*^{-1} \|\tilde{T}_h w_h - \tilde{T}_h v_h\|_1 . \tag{3.4}$$

Proof. Let us prove (3.4). We set $z_i = w_i\lambda'(t_i)$, $y_i = v_i\lambda'(t_i)$, $i=1,2,\dots n-1$, and introduce a new operator:

$$\bar{T}_h z_i = \tilde{T}_h(z_i/\lambda'(t_i)) , \quad i=1,2,\dots,n-1 .$$

Then the Frechet derivative of $\bar{T}_h$ satisfies:

$$\|(\bar{T}_h'(z_h))^{-1}\|_1 \le 1/c_* ,$$

see the proof of Theorem 2.1. Futhermore:

$$\|z_h - y_h\|_1 \le c_*^{-1} \|\bar{T}_h z_h - \bar{T}_h y_h\|_1 ,$$

and (3.4) is immediate. The inequalities

$$\tilde{T}_h(\bar{u}e_h) \ge 0 \ge \tilde{T}_h(\underline{u}e_h)$$

complete the proof. □

Let us now consider the consistency error of the operator $\tilde{T}_h$:

$$\tilde{r}_i = (\tilde{T}\tilde{u}_\varepsilon)(t_i) - \tilde{T}_h\tilde{u}_\varepsilon(t_i) , \quad i=1,2,\dots,n-1 .$$

As before, we set:

$$\tilde{r}_i = \tilde{r}_i'' + \tilde{r}_i' .$$

where

$$\tilde{r}_i'' = \varepsilon \left[D_h''\tilde{u}_\varepsilon(t_i) - (\rho(t)\tilde{u}_\varepsilon'(t))'_{t=t_i} \right] ,$$

$$\tilde{r}_i' = f(\tilde{u}_\varepsilon)'(t_i) - \tilde{D}_h'\tilde{u}_\varepsilon(t_i) .$$

Additionally to (1.9) and (2.2) we shall assume that:

$$|\lambda''(t)| \le M , \quad t \in I , \tag{3.5}$$

and that λ''' is piecewise continuous in I, in such a way that:

$$\lambda \in C^3(t_{i-1}, t) , \quad i=1,2,\dots,n , \tag{3.6}$$

$$|\lambda'''(t)| \le M, \quad t \in (t_{i-1}, t_i), \quad i=1,2,\dots,n . \tag{3.7}$$

Let

$$\tilde{J}(i,j,k) = \int_{t_i}^{t_j} (s - t_j)^{k-1} \tilde{u}_\varepsilon^{(k)}(s)ds$$

and let $g_{u,i}$, $g_{v,i}$ have the same meaning as in Section 1. We can easily prove the following lemma, which corresponds to Lemma 2.1 in the direct approach.

Lemma 3.1 *Let (1.3), (1.15) and (3.6) holds. Then for $i=1,2,\dots,n-1$ we have:*

$$\tilde{r}_i'' = \varepsilon h\Big[((\rho\tilde{u}_\varepsilon')''(\sigma_i^+) - (\rho\tilde{u}_\varepsilon')''(\sigma_i^-))/8 +$$

$$+ (\rho(t_{i+1/2})\tilde{u}_\varepsilon'''(\Theta_i^+) - \rho(t_{i-1/2})\tilde{u}_\varepsilon'''(\Theta_i^-))/48 \Big] ,$$

where σ_i^-, $\Theta_i^- \in (t_{i-1}, t_i)$, σ_i^+, $\Theta_i^+ \in (t_i, t_{i+1})$,

and

$$\tilde{r}_i' = g_{u,i}\tilde{J}(i,i+1,2)/h + g_{v,i}\tilde{J}(i,i-1,2)/h +$$

$$+\, h^{-1}\left[\int_{\tilde{u}(t_i)}^{\tilde{u}_\varepsilon(t_{i+1})} (g_{u,i} - g_u(s,\tilde{u}_\varepsilon(t_i)))\,ds \;+\right.$$

$$\left. +\int_{\tilde{u}(t_{i-1})}^{\tilde{u}_\varepsilon(t_i)} (g_{v,i} - g_v(\tilde{u}_\varepsilon(t_i),s))\,ds\right].$$

Theorem 3.2 *Let* (1.3), (1.4), (1.14-17), (3.6) *and* (3.7) *hold. Then we have the convergence* (1.11).

Proof. From (3.4) it follows:

$$\sum_{i=1}^{n-1} \lambda'(t_i)\,|w_{\varepsilon,i} - \tilde{u}_\varepsilon(t_i)| \le c_*^{-1}\,\|\tilde{r}_h\|_1 \le M_\varepsilon .$$

Multiply this inequality by h, use $\tilde{u}_\varepsilon(t_i) = u_\varepsilon(x_i)$ and

$$|h\lambda'(t_i) - \bar{h}_i| \le Mh^3$$

(which is valid because of (3.6), (3.7)), and obtain

$$\|w_{\varepsilon,h} - u_{\varepsilon,h}\|_h \le M_\varepsilon h + Mh^2 ,$$

(since $|w_{\varepsilon,i} - u_\varepsilon(x_i)| \le M$). □

Finally we can prove:

Theorem 3.3 *Let the conditions of Theorem* 3.2 *hold and let the following inequalities be satisfied:*

(3.8) $$\varepsilon|\rho^{(k)}(t)| \le M ,\quad k=0,1,2 ,$$

(3.9) $$|\tilde{u}_\varepsilon^{(k)}(t)| \le M ,\quad k=1,2,3 ,$$

where $t \in (t_{i-1}, t_i)$, $i=1,2,\ldots,n$. *Then the uniform convergence* (1.13) *holds.*

From this theorem we can conclude how to choose λ. This can be used in the direct approach as well.

Let S denote a layer of u_ε. Very often u'_ε behaves in the following way:

$$u'_\varepsilon(x) \sim v'_\varepsilon(x) \ , \ x \in S \ ,$$

where v_ε is the corresponding layer function. If we take

$$\lambda(t) = v_\varepsilon^{-1}(t) \quad \text{for} \quad t \in \lambda^{-1}(S) \ ,$$

then (3.9) follows for k=1 and $x \in S$. With such a choice of λ we can expect that (3.9) holds for k=2,3 as well. On the other hand, outside of layers we need (2.2), (3.5), (3.7), and in these parts λ can be a suitable polynomial.

4. Two special problems

4.1 A non-turning point problem

In this section we shall consider the problem (1.1), (1.2) in the case when

$$-b(u) \geq b_* > 0, \quad u \in W. \tag{4.1}$$

The proof of the following theorem can be found in [23], cf. [5].

Theorem 4.1. *Let* (1.3), (1.4) *and* (4.1) *hold. Then for* $x \in I$ *we have:*

$$|u_\varepsilon^{(k)}(x)| \leq M(1 + \varepsilon^{-k} v_\varepsilon(x)), \quad k=0,1,2,3 \ , \tag{4.2}$$

$$v_\varepsilon(x) = \exp(-b_\circ x/\varepsilon) \ .$$

Thus, in this case u_ε has a boundary layer at x=0. Let us apply the direct approach. First we have to choose λ according to (4.2). In view of the discussion at the end of Section 3, we take:

$$\lambda(t) = \begin{cases} \omega(t) \ , \ t \in [0,\alpha] \ , \\ \pi(t) := \delta(t-\alpha)^3 + \omega''(\alpha)(t-\alpha)^2/2 + \omega'(\alpha)(t-\alpha) + \omega(\alpha), \ t \in [\alpha,1], \end{cases} \tag{4.3}$$

where $\alpha \in (0,1)$ is given, δ is determined from $\pi(1)$=1, and $\omega(t)$ corresponds to the inverse of $v_\varepsilon(x)$. Essentially the following type of function was introduced in [2]:

$$\omega(t) = \omega_1(t) := -\beta\varepsilon\ln(1 - t/\gamma) = \beta\varepsilon\ln(1 + t/(\gamma - t)),$$

where β is a positive parameter and

$$\gamma = \alpha + \varepsilon^{1/3} .$$

Similar logarithmic functions were used in [7-9]. However, it was shown in [17] that for this type of layer a certain approximation to $\omega_1(t)$ would suffice, e.g.:

$$\omega(t) = \omega_2(t) := \beta\ \varepsilon\ t/(\gamma - t) . \tag{4.4}$$

For the reasons of simplicity, we shall consider here functions of $\omega_2(t)$ - type only. Functions of this type were used in [17-25].

It is obvious that $\lambda \in C^2(I)$. From now on we shall assume that the parameter β in (4.4) is given in such a way that

$$\delta \ge 0 .$$

It is easy to see that such a positive β, independent of ε, exists. This implies:

$$0 < \lambda^{(k)}(t), \quad k=1,2, \quad t \in I .$$

Moreover, because of the special choice of γ we have

$$\lambda^{(k)}(t) \le M, \quad k=1,2, \quad t \in I ,$$

hence the conditions (2.2), (2.3) are satisfied.

Note the following property of λ, which we shall use later

$$\exp(-b_\circ\lambda(t)/\varepsilon) \le M\exp(-M/(\gamma - t)), \quad t \in (0,\alpha) .$$

Theorem 4.2 *Let* (1.3), (1,4), (4.1), (4.1), (1.14-17) *hold and let the mesh* I_h *be given by* (1.8), (4.3), (4.4). If

$$n \ge m\ln\varepsilon , \tag{4.6}$$

with an appropriate constant $m > 0$, *independent of* h *and* ε, *then we have* (1.13).

Proof. According to Theorem 2.3 we only have to prove (2.6-8). For instance, let us prove the first inequality from (2.8). The other inequalities can be proved analously. We shall use the standard technique from [2], [17-21], [23-25], which consists of the following three steps (for $i=1,2,\dots,n-1$):

$$1^\circ \quad t_{i-1} \ge \alpha - \varepsilon^{1/3} ,$$

$$2^\circ \quad t_{i-1} \le \alpha - 3h ,$$

$$3^\circ \quad \alpha - 3h < t_{i-1} < \alpha - \varepsilon^{1/3} .$$

Thus, since $h_i \le h_{i+1}$, $i=1,2,\dots n-1$, we have to prove

$$(4.7) \qquad V := (h_{i+1}/h_i) \int_{x_{i-1}}^{x_i} (s-x_{i-1}) \left| u''_\varepsilon(s) \right| ds \le Mh^2 , \qquad i=1,2,\dots n-1 .$$

Because of (4.2) we have

$$V \le Mh^2_{i+1}(1 + \varepsilon^{-2})v_\varepsilon(\lambda(t_{i-1}))) .$$

We shall use this inequality in steps 1° and 2° .

In the step 1° it holds that

$$V \le Mh^2_{i+1}(1 + \varepsilon^{-2}v_\varepsilon(\lambda(\alpha - \varepsilon^{1/3})))$$

and (4.7) follows from (2.4) and (4.5).

In the step 2° we use (4.5) and

$$h_{i+1} \le Mh\varepsilon(\gamma - t_{i+1})^{-2} \le Mh\varepsilon(\gamma - t_{i-1})^{-2} ,$$

(since in this case $\alpha - t_{i+1} \ge (\alpha - t_{i-1})/3$, and (4.7) is proved again.

Finally, in the step 3° we have

$$V \le M(h^2_{i+1} + h_{i+1}\varepsilon^{-1}v_\varepsilon(x_{i-1})) \le$$

$$\le M(h^2 + h\varepsilon^{-1}v_\varepsilon(\lambda(\alpha - 3h))) \le$$

$$\le M(h^2 + h\varepsilon^{-1} \exp(-Mn)) ,$$

and (4.7) follows because of (4.6) and $\varepsilon^{1/3} < 3h$. □

Let us now consider the indirect approach. Besides (1.9), the function λ should satisfy (2.2), (3.5-7). Because of that we take λ from (4.3) with

$$(4.8) \qquad \omega(t) = \omega_3(t) := \beta\varepsilon t/(\alpha + \varepsilon^{1/4} - t),$$

where

$$\alpha = j/n \quad \text{for some} \quad j \in \{1,2,\dots n-1\},$$

so that (3.6) holds. Again, the positive parameter β should be chosen so that $\delta \ge 0$. The same function λ was used in [23], [24].

Theorem 4.3 *Let* (1.3), (1.4), (4.1), (1.14-17) *hold and let the mesh* I_h *be given by* (1.8), (4.3), (4.8). *Then we have* (1.13).

Proof. Because of Theorem 3.3 it is sufficient to prove (3.6) and (3.9). But these two inequalities have already been proved in [23]. □

Let us mention that a more general problem has been considered in [23] - the case when $b = b(x,u)$, and that the linear convergence uniform in ε has been proved in the equidistant norm $\|\cdot\|_h = h\|\cdot\|_1$.

4.2 A boundary shock problem

Let us now consider the case when

$$b(u) = u\, b_1(u), \quad c(x,u) = u\, c_1(x,u)$$

$$b^* \ge b_1(u) \ge b_* > 0, \quad u \in W,$$

$$c^* \ge c_1(x,u) \ge c_* \ge 0, \quad x \in I, \quad u \in W,$$

$$U_0 = 0, \quad U_1 > \{b^*c^* + [b^*c^*(b^*c^* - b_*c_*)]^{1/2}\}/(b^*b_*).$$

These conditions together with (1.3) and (1.4) guarantee the following estimates of the derivatives of the exact solution:

$$|u_\varepsilon^{(k)}(x)| \le M(1 + \varepsilon^{-k}\exp(-m_0 x/\varepsilon)), \quad x \in I, \quad k=1,2,3,$$

with a positive constant m_0 independent of ε. For the proof see [24], where the indirect approach was used. Here the solution behaves in the same way as in the case of the non-turning point problem, thus the same functions λ can be used, and the results analogous to Theorem 4.2 and 4.3 can be proved.

5. Concluding remarks and numerical results

Let us consider the problem from [14], [23]:

P1. $$-\varepsilon u'' - \exp(u)u' + ((\pi/2)\sin(\pi x/2))\exp(2u) = 0,$$

$$u(0) = u(1) = 0 \ ,$$

the solution of which is given by

$$u_\varepsilon(x) = y_\varepsilon(x) + O(\varepsilon) \ ,$$

where

$$y_\varepsilon(x) = -\ln[(1 + \cos(\pi x/2))\,(1 - (1/2)\exp(-x/(2\varepsilon)))] \ .$$

We shall compare our numerical results with $y_\varepsilon(x)$. Let

$$E_\infty = \|y_{\varepsilon,h} - w_{\varepsilon,h}\|_\infty \ , \quad E_h = \|y_{\varepsilon,h} - w_{\varepsilon,h}\|_h \ .$$

We shall use the function λ from (4.3) with (4.4) and (4.8) in the direct and indirect approaches, respectively. In both approaches we take

$$\alpha = 0.5 \ , \quad \beta = 1 \ ,$$

getting 24 - 33 % of the mesh points in the layer, (in dependence of ε). The effect of changing α and β can be seen in [17], [23], [25], for instance. Both *EO* and *LF* schemes will be used, the latter with

$$B = 1 \ .$$

TABLE 5.1 P1, direct approach, EO scheme

n	ε	10^{-3}	10^{-8}	10^{-9}
50	E_∞	8.27-2	8.49-2	8.56-2
	E_h	5.67-2	5.78-2	5.79-2
100	E_∞	4.42-2	4.48-2	4.49-2
	E_h	3.04-2	3.06-2	3.06-2

TABLE 5.2 P1, direct approach, LF scheme

n	ε	10^{-3}	10^{-6}	10^{-9}
50	E_∞	8.82-2	0.104	0.105
	E_h	5.89-2	5.99-2	6.00-2
100	E_∞	5.41-2	5.56-2	5.60-2
	E_h	3.10-2	3.11-2	3.12-2

TABLE 5.3 P1, indirect approach, EO scheme

n	ε	10^{-3}	10^{-6}	10^{-9}
50	E_∞	8.88-2	8.83-2	8.89-2
	E_h	5.99-2	6.03-2	6.03-2
100	E_∞	4.78-2	4.64-2	4.67-2
	E_h	3.22-2	3.19-2	3.20-2

TABLE 5.4 P1, indirect approach, LF scheme

n	ε	10^{-3}	10^{-6}	10^{-9}
50	E_∞	0.108	0.107	0.108
	E_h	6.23-2	6.26-2	6.26-2
100	E_∞	5.91-2	5.72-2	5.78-2
	E_h	3.28-2	3.25-2	3.25-2

The numerical results show more than our theory gives: the pointwise convergence uniform in ε can be observed. We can see that there is no big difference between the results of Tables 4.1-4. However, the direct approach gives somewhat better result. This confirms that the condition (4.6) of Theorem 4.2 is not essential for the proof of (1.3). It is introduced for technical reasons only - in the step 3° of the proof. (Note that the proof of Theorem 4.2 could be simpler - because of (4.6) we could use two steps only: 1° $t_{i-1} \geq \alpha - 3h$ and 2° $t_{i-1} \leq \alpha - 3h$. By the proof which is given here we want to show that the condition (4.6) is needed in the step 3° only).

On the other hand, EO scheme is better than LF, which is not surprising, since in this case EO scheme reduces to the standard upwind scheme which is here more natural than LF scheme.

Similar conclusions hold in the case of the boundary shock problem, cf. [24].

Comparing the two approaches we can conclude that the proofs of the uniform convergence are simpler and easier in the case of the indirect approach. In particular, there is no need for some artifical conditions of (4.6) - type. However, the direct approach seems simpler for coding and it uses simpler functions λ. We illustarte this by the functions which should be used in the case of two boundary layers:

$$(5.1) \qquad \lambda(t) = \begin{cases} \omega_2(t), & t \in [0,\alpha] \\ \pi_1(t), & t \in [\alpha, 1/2] \\ 1 - \lambda(1-t), & t \in [1/2, 1] \end{cases},$$

where $\alpha \in (0, 1/2)$ and $\pi_1(t)$ is a third order polynomial, such that $\lambda \in C^2[0,1/2]$ and $\pi_1(1/2) = 1/2$. It holds that $\lambda \in C^1[0,1]$ and λ'' is discontinuous at $t=1/2$, but the direct approach allows this since (2.3) holds for $t \in I \setminus \{1/2\}$. In the indirect approach $\omega_2(t)$ should be replaced by $\omega_3(t)$, and we should use a more complicated polynomial, which connects $\omega_3(t)$ and $1 - \omega_3(1-t)$ in such a way that (3.5-7) hold.

Similar facts hold in the case of function

$$\lambda(t) = \begin{cases} \omega_2(t), & t \in [0,\alpha] \\ \pi_1(t), & t \in [\alpha, 1] \\ -\lambda(-t), & t \in [-1,0] \end{cases},$$

which should be used when the interval is $[-1,1]$ and there is an interior layer at $x=0$.

Let us use the function (5.1) to solve the following problem by the direct approach:

P2. $\qquad -\varepsilon u'' + u\,u' + u + s(x) = 0, \quad u(0) = U_0, \quad u(1) = U_1,$

where $s(x)$ and U_0, U_1 are given so that the solution reads:

$$u_\varepsilon(x) = -\exp(-x/\varepsilon) + \exp((x-1)/\varepsilon) .$$

In Table 5.5 we give the results obtained by EO scheme for α =o.25 and β=1. Let E_∞ and E_h be as before, except that here we take u_ε instead of y_ε. Note that here the error E_h decreases together with ε.

TABLE 5.5 P2, direct approach, EO scheme n=100

ε:	10^{-3}	10^{-6}	10^{-9}
E_∞	1.12-2	3.68-2	4.16-2
E_h	1.32-3	9.33-5	6.21-6

Of course, in general it is not easy to know in advance where the layers are. In the case when locations of the layers are not known, one should apply a stable equidistant scheme to locate the layers. After that, a suitable mesh generating function could be introduced.

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Rezime

METODI GENERISANJA MREŽA ZA NUMERIČKO REŠAVANJE KVAZILINEARNIH SINGULARNIH PERTURBACIONIH PROBLEMA

Kvazilinearni singularno perturbovani konturni problemi se rešavaju numerički korišćenjem diferencnih šema na specijalnim neekvidistantnim mrežama. Mreže su generisane pogodnim funkcijama koje preraspoređuju ekvidistantne tačke. Dva slična pristupa generisanju mreža su uporedena. "Uniformna" konvergencija je dokazana za dva specijalna tipa problema. Numerički rezultati ilustruju efikasnost metoda.

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Rezime

METODI GENERISANJA MREŽE ZA NUMERIČKO REŠAVANJE KVAZILINEARNIH SINGULARNIH PERTURBACIONIH PROBLEMA

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EXAMPLE OF A CONTINUOUS NON-MARKOVIAN PROCESS $X(t)=W_1(t)+\varphi(t)W_2(t)$ OF MULTIPLICITY TWO

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Abstract

The continuous process $X(t)= W_1(t) + \varphi(t)W_2(t)$, $0\le t\le 1$, is considered, where $W_1(t)$ and $W_2(t)$ are independent Wiener processes and $\varphi(t)$ is a version of the Cantor distribution of function. The multiplicity of the non-Markovian process $X(t)$ is two. The proper canonical representation of $X(t)$ is also given.

0. Let $\{X(t),\ t\ge 0\}$ be a mean-square continuous and purely non-deterministic process. The proper canonical (or Hida-Cramer) representation of $\{X(t)\}$ is

$$X(t) = \sum_{1}^{N} \int_{0}^{t} g_n(t,u)\, dZ_n(u) , \tag{1}$$

where

- the so-called innovation processes $\{Z_n(t), t\ge 0\}$, $n=1,\dots,N$ (N may be ∞) are mutually orthogonal wide-sense martingals,
- the mean-square linear closure of $\{X(u),\ u\le t\}$: $\mathcal{H}(X;t)$ coincides with $\sum_{1}^{N} \oplus\ \mathcal{H}(Z_n;t)$ for all t ($\mathcal{H}(X) = \overline{\bigvee_t \mathcal{H}(X;t)}$ is the Hilbert space with the inner product $\langle X,Y\rangle = EXY$),

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- the measures $dF_n(t) = \|dZ_n(t)\|^2$ are ordered by the absolute continuity

$$dF_1(t) \gtrsim dF_2(t) \gtrsim \dots \gtrsim dF_N(t) .$$

Let ρ_n be the class of measures equivalent by the absolute continuity to $dF_n(t)$. The chain

$$\rho_1 \gtrsim \rho_2 \gtrsim \dots \gtrsim \rho_N \tag{2}$$

is the spectra type of $\{X(t)\}$ and N is the spectral multiplicity of $\{X(t)\}$. The correlation function of $\{X(t)\}$ uniquely determines the spectral type. The main result of [1] is that for an arbitrary chain (2) there exists a continuous process with the spectral type (2).

In connection with this result interesting are the examples of univariate continuous processes with $N \geq 2$. It was shown in [3] that the M-ple Markovian process $\{Y(t)\}$ with the proper Goursat representation

$$Y(t) = \sum_1^M f_n(t)\ Z_n(t)$$

has the multiplicity $N \in \{1,\dots,M\}$ depending of some continuity and smoothness properties of the functions $f_n(t)$. For example, ([3], [2]) the process $\{Y(t)\}$ with the representation

$$Y(t) = W_1(t) + f(t)W_2(t) , \tag{3}$$

where $\{W_1(t)\}$ and $\{W_2(t)\}$ are mutually orthogonal Wiener processes, has the spectral type

$$dt \sim dt ,$$

if $f(t)$ is continuous but not absolutely continuous in any interval. In this case, $\{Y(t)\}$ is the 2-ple Markovian process and (3) is its proper canonical representation.

1. In this note we shall give the example of a continuous non-Markovian process $\{X(t),\ 0 \leq t \leq 1\}$ with the representation

$$X(t) = W_1(t) + \varphi(t)\ W_2(t) , \tag{4}$$

where $\{W_1(t)\}$ and $\{W_2(t)\}$ are multually orthogonal Wiener processes. The function $\varphi(t)$ is a modification of the distribution function on the Cantor ternary set.

We define $\varphi(t)$, $0 \le t \le 1$, in the following way: $\varphi(0) = 0$, $\varphi(1) = 1$; $\varphi(t) = \frac{2}{3}\,\varphi(0) + \frac{1}{3}\,\varphi(1) = \frac{1}{3}$, $\frac{1}{3} \le t \le \frac{2}{3}$; $\varphi(t) = \frac{2}{3}\,\varphi(0) + \frac{1}{3}\,\varphi(\frac{1}{3}) = \frac{1}{9}$, $\frac{1}{9} \le t \le \frac{2}{9}$; $\varphi(t) = \frac{2}{3}\,(\frac{2}{3}) + \frac{1}{3}\,\varphi(1) = \frac{5}{9}$, $\frac{7}{9} \le t \le \frac{8}{9}$; ... and so on.

In such a way $\varphi(t)$ is a continuous distribution function increasing only on the Cantor set C. Let us denote by L and R; $L, R \subset C$; the countable sets of left and right, respectively, end-points of removed open intervals in the definiton of C. According to the definition of $\varphi(t)$, it is easy to see that for $t' \in L$

$$(5) \qquad \lim_{h \downarrow 0} \frac{[\varphi(t') - \varphi(t'-h)]^2}{h} = +\infty$$

In such a way, for the process $\{X(t)\}$ defined by (4), we have

$$(6) \qquad \underset{h \downarrow 0}{\text{l.i.m.}}\ \frac{X(t') - X(t'-h)}{\varphi(t') - \varphi(t'-h)} = W_2(t') = \dot{X}_{\varphi}(t') .$$

Indeed, $(\Delta\varphi = \varphi(t') - \varphi(t'-h))$,

$$\left\| \frac{X(t') - X(t'-h)}{\Delta\varphi} - W_2(t') \right\|^2 = \frac{1}{(\Delta\varphi)^2} \Big\| W_1(t') - W_1(t'-h) +$$

$$+ \varphi(t'-h)\,[W_2(t') - W_2(t'-h)] \Big\|^2 = \frac{1}{(\Delta\varphi)^2}\,[1 + \varphi^2(t'-h)] \longrightarrow 0 \quad h \downarrow 0.$$

Let us denote by P_t the projection operator onto $\mathcal{H}(X,t)$ and consider the wide-sense martingal $\{Z_1(t),\ 0 \le t \le 1\}$ defined by

$$Z_1(t) = P_t X(1).$$

Evidently, $\{Z_1(t)\}$ is an innovation process of $\{X(t)\}$.

$\{X(t)\}$ being continuous, the space $\mathcal{H}(X)$ is separable. For arbitrary t, $0 < t < 1$, consider the partition $\frac{k}{2^n}\,t$, $k=1,\dots,2^n$ of $[0,t]$ for $n=1,2,\dots$ and the mean-square linear closure $\mathcal{H}_n(X;t)$ of $\{X(\frac{k}{2^n}\,t),\ k=1,\dots,2^n\}$. By the separability of $\mathcal{H}(X;t)$ and $\mathcal{H}_1(X;t) \subset \mathcal{H}_2(X;t) \subset \dots$ we conclude that

$$\mathcal{H}(X;t) = \overline{\bigvee_n \mathcal{H}_n(X;t)} .$$

Let us denote by P_{nt} the projection operator onto $\mathcal{H}_n(X;t)$. We have

(7) $$Z_1(t) = \operatorname*{l.i.m.}_{n\to\infty} P_{nt}X(t) . \qquad (7)$$

Simplifyng the notations by $h = \dfrac{1}{2^n}t$, we consider the projection of $X(1)$ on random variables $X(t)$, $X(t-h)$, $X(t-2h)$,...:

$$P_{nt}X(1) = aX(t) + bX(t-h) + cX(t-2h) + \dots$$

It is easy to see that

$$\langle X(1) - [aX(t) + bX(t-h)],\ X(u)\rangle = 0 \quad \text{for all} \quad u \le t - h ,$$

if $\quad a = \dfrac{1 - \varphi(t-h)}{\Delta\varphi} , \quad b = \dfrac{\varphi(t) - 1}{\Delta\varphi} .$

Rewrite

$$aX(t) + bX(t-h) = X(t) + [1 - \varphi(t)]\,\frac{X(t) - X(t-h)}{\Delta\varphi} .$$

By (5), we have for $t' \in L$

(8) $$Z_1(t') = X(t') + [1 - \varphi(t')]\dot{X}_{\varphi}(t') = W_1(t') + W_2(t')$$

Put

(9) $$Z_2(t') = W_1(t') - W_2(t') .$$

We have $\langle Z_1(t'), Z_2(t')\rangle = 0$. Using (6) and (7), we write

(10) $$X(t') = \frac{1 + \varphi(t')}{2}\, Z_1(t') + \frac{1 - \varphi(t')}{2}\, Z_2(t') .$$

We conclude from (6) that $W_2(t') \in \mathcal{H}(X;t')$ and from (8) and (9) we conclude that $Z_1(t')$, $W_1(t')$, $Z_2(t') \in \mathcal{H}(X;t')$. In this way (10) is the proper canonical representation of $\{X(t)\}$ at the points $t' \in L$.

Let (t', t'') be a removed interval and let $t \in (t', t'']$. We verify, by simple calculation, that

$$\left\langle X(1) - \left[Z_1(t') + \frac{1 + \varphi(t')}{1 + \varphi^2(t')}\,(X(t) - X(t'))\right],\ X(u)\right\rangle = 0 \quad \text{for all} \quad u \le t .$$

This way

(11) $$Z_1(t) = Z_1(t') + \alpha(t')(X(t) - X(t')), \quad \alpha(t') = \frac{1 + \varphi(t')}{1 + \varphi^2(t')} ,$$

for $t \in [t', t'']$ with $Z_1(t'+0) = Z_1(t')$ and $Z_1(t''-0) = Z_1(t'')$.

Let us remark that $Z_1(t''+0) \neq Z_1(t'')$. Indeed, consider the sequence of intervals (t'_n, t''_n) such that $t'_n \downarrow t''$, $n \to \infty$. Then $Z_1(t'_n) = W_1(t'_n) + W_2(t'_n) \to$ $\to W_1(t'') + W_2(t'')$, in the mean-square as $n \to \infty$ and

$$\lim_{n\to\infty} \|Z_1(t'_n) - Z_1(t'')\|^2 = \|[W_1(t'') + W_2(t'')] - [Z_1(t') + \alpha(t')\, X(t'') - X(t')]\|^2 =$$

$$\frac{[1-\varphi(t')]^2}{1+\varphi^2(t')}\,(t''-t') \neq 0$$

Rewrite (11) in the form

$$X(t) = X(t') + \frac{1}{\alpha(t')}\,(Z_1(t) - Z_1(t'))$$

or

$$(12) \qquad X(t) = \frac{1+\varphi^2(t')}{1+\varphi(t')}\,Z_1(t) - \frac{(1-\varphi(t'))^2}{2(1+\varphi(t'))}\,Z_1(t') + \frac{1-\varphi(t')}{2}\,Z_2(t').$$

(12) is the proper canonical representation of $\{X(t)\}$ for $t \in (t', t'']$.

There remains to find the proper canonical representation of $\{X(t)\}$ for $t = s \in C\setminus(L \vee R)$. Consider two sequences of removed intervals: the sequence $(\underline{t}'_n, \underline{t}''_n)$ such that $\underline{t}''_n \uparrow s$ (then in fact $\underline{t}'_n \uparrow s$, because $\underline{t}''_n - \underline{t}'_n \to 0$) and the sequence $(\bar{t}'_n, \bar{t}''_n)$ such that $\bar{t}'_n \downarrow s$. It follows from the continuity of $\{X(t)\}$ and $\{W_1(t)\}$ that

$$\underset{n\to\infty}{\mathrm{l.i.m.}}\ X(\underline{t}'_n) = X(s) = \underset{n\to\infty}{\mathrm{l.i.m.}}\ \bar{X}(t'_n)$$

or

$$\underset{n\to\infty}{\mathrm{l.i.m.}}\left[\frac{1+\varphi(\underline{t}'_n)}{2}\,Z_1(\underline{t}'_n) + \frac{1-\varphi(\underline{t}'_n)}{2}\,Z_2(\underline{t}'_n)\right] = X(s) =$$

$$= \underset{n\to\infty}{\mathrm{l.i.m.}}\left[\frac{1+\varphi(\bar{t}'_n)}{2}\,Z_1(\bar{t}'_n) + \frac{1-\varphi(\bar{t}'_n)}{2}\,Z_2(\bar{t}'_n)\right]$$

and

$$\|Z_1(\bar{t}'_n) - Z_1(\underline{t}'_n)\|^2 = \|[W_1(\bar{t}'_n) \pm W_2(\bar{t}'_n)] - [W_1(\underline{t}'_n) \pm W_2(\underline{t}'_n)]\|^2 =$$

$$= 2(\bar{t}'_n - \underline{t}_n) \to 0, \quad n \to \infty\,.$$

We conclude that

$$X(s) = \frac{s+\varphi(s)}{2}\,Z_1(s) + \frac{1-\varphi(s)}{2}\,Z_2(s), \quad Z_i(s) = W_1(s) \pm W_2(s), \quad i=1,2,$$

is the proper canonical representation of $\{X(t)\}$.

Let us summarize:

The process $\{X(t),\ 0 \le t \le 1\}$ *given by* (4) *has the following proper canonical representation*

$$X(t) = \begin{cases} \dfrac{1+\varphi(t)}{2}\, Z_1(t) + \dfrac{1-\varphi(t)}{2}\, Z_2(t), & t \in C\setminus R \\[2ex] \dfrac{1+\varphi^2(t)}{1+\varphi(t)}\, Z_1(t) - \dfrac{(1-\varphi(t'))^2}{2(1+\varphi(t'))}\, Z_1(t') + \dfrac{1-\varphi(t')}{2}\, Z_2(t'), & t \in C^c \vee R \end{cases}$$

where $t' = t'(t)$ *is the left-end point of the removed interval* (t', t''), $t \in (t', t'']$.

The innovation processes $\{Z_1(t),\ 0 \le t \le 1\}$ $i=1,2$ *are*

$$Z_1(t) = \begin{cases} W_1(t) + W_2(t), \quad t \in C\setminus R \\[2ex] \dfrac{1+\varphi(t)}{1+\varphi^2(t)}[W_1(t) + \varphi(t)W_2(t)] + \dfrac{\varphi(t')(1-\varphi(t'))}{1+\varphi^2(t')}\, W_1(t') + \dfrac{1-\varphi(t')}{1+\varphi^2(t')}\, W_2(t'), \\[2ex] t \in C^c \vee R, \end{cases}$$

$$Z_2(t) = \begin{cases} W_1(t) - W_2(t), & t \in C\setminus R \\ W_1(t') - W_2(t'), & t \in C^c \vee R. \end{cases}$$

Let us make two remarks at the end.

The process $\{X(t)\}$ is not Markovian. According to the definition in [3], a process $\{Y(t)\}$ is M-ple Markovian if for all $a \le b$ set $\{P_a Y(t), t \ge b\}$ contains exactly M linearly independent elements. In our example, the set $\{P_a X(t),\ t \ge a\}$ for $a = t'$ contains two linearly independent elements $W_1(t'), W_2(t')$ and for $a = t \in (t', t'']$ contains three linearly independent elements $\dot{W}_1(t), W_1(t'), W_2(t')$.

Concerning the spectral type

$$\rho_1 > \rho_2$$

of the process $\{X(t)\}$, the measure $dF_2(t) = d\|Z_2(t)\|^2$ belonging to ρ_2 is a discrete measure on the set R, the measure $dF_1(t) = d\|Z_1(t)\|^2$ belonging to ρ_1 is the sum of a measure equivalent to the ordinary Lebesgue measure dt and a measure equivalent to $dF_2(t)$.

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Rezime

PRIMER NEPREKIDNOG NE-MARKOVSKOG PROCESA $X(t) = W_1(t) + \varphi(t)W_2(t)$ MULTIPLICITETA DVA

Posmatra se neprekidni proces $X(t) = W_1(t) + \varphi(t)W_2(t)$, $0 \le t \le 1$, gde su $W_1(t)$ i $W_2(t)$ nezavisni Vinerovi procesi i $\varphi(t)$ je jedna varijanta Kantorove funkcije raspodele. Multiplicitet ne-markovskog procesa $X(t)$ je dva. Takođe je data čisto kanonička reprezentacija za $X(t)$.

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A CLASS OF GENERALIZED RANDOM PROCESSES WITH VALUES IN $L^2(\Omega)$

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Abstract

The structural theorems for generalized random processes from $L(\mathcal{A}_k, Z)$ and $Y_{\cdot}^{-k}$ are given.

1. Introduction

In [11] Zemanian introduced the space $\mathcal{A}$, the space of test functions and its dual space $\mathcal{A}'$. Using his ideas we construct a scale of spaces $\mathcal{A}_k$, where k is an integer, whose elements have an orthonormal expansion. Next, we define a generalized random process (g.r.p.) as a continuous linear mapping from $\mathcal{A}_k$ to Z - a separable Hilbert space of random variables with finite second moments. We denote the space of all g.r.p. by $L(\mathcal{A}_k, Z)$. In the definition of g.r.p. we follow [4]. This definition is different from the definitions given in [1,2,7,8,9,10]. In Section 3.2. we construct the space Y^{-k}, where k is an integer, a subspace of $L(\mathcal{A}_k, Z)$. Giving the structural theorems for elements in Y^{-k} and $L(\mathcal{A}_k, Z)$, we establish the relation between them.

Since elements from the spaces $\mathcal{A}_k$ have orthonormal expansions, this enables us to give simpler structural theorems than in [4], where Sobolev

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Key words: generalized random process, the Zemanian space $\mathcal{A}$

spaces were observed. The proofs of Theorems 4.1. and 4.2. are given following [4]. The given structural theorems can be applied in solving some classes of stochastic differential equations similar as in [7].

2. Spaces $\mathcal{A}_k$ and $\mathcal{A}'_k$

2.1. We shall follow the notation as in [11, Ch. 9.]. Let I be an open interval in the set of real numbers $\mathbb{R}$, $L^2(I)$ the space of the equivalence classes of square integrable functions with values in the set of complex numbers $\mathbb{C}$. The norm in $L^2(I)$ is defined by

$$\|f\|_0 = \left(\int_I |f(t)|^2 dt\right)^{1/2}$$

Denote by $C^\infty(I)$ the set of infinitely differentiable (smooth) functions and by $\mathbb{N}_0$ and $\mathbb{N}$ sets $\{0,1,2,\dots\}$, $\{1,2,\dots\}$, respectively.

Let $\mathcal{R}$ be a linear differential self-adjoint operator of the form

$$\mathcal{R} = \theta_0 D^{n_1}\theta_1 \dots D^{n_\nu}\theta_\nu ,$$

where $D=d/dx$, n_k, $k=1,2,\dots,\nu$, are non-negative integers θ_k, $k=0,1,\dots,\nu$, smooth complex functions with no zeros on I. Suppose that there exists a sequence of real numbers $\{\lambda_n,\ n \in \mathbb{N}_0\}$ and a sequence $\{\psi_n,\ n \in \mathbb{N}_0\}$ of smooth functions in $L^2(I)$ such that $|\lambda_n| \to \infty$ for $n \to \infty$ and

$$\mathcal{R}\psi_n = \lambda_n \psi_n \qquad n \in \mathbb{N}_0 .$$

Furthermore, suppose that $\{\psi_n,\ n \in \mathbb{N}_0\}$ forms an orthonormal system (o.n.s.) in $L^2(I)$. We can enumerate λ_n and ψ_n so that $|\lambda_0| \le |\lambda_1| \le \lambda_2| \le \dots$

Put

$$\tilde{\lambda}_n = \begin{cases} \lambda_n & \text{if } \lambda_n \neq 0 \\ 1 & \text{if } \lambda_n = 0 \end{cases} \qquad n \in \mathbb{N}_0$$

$\{\tilde{\lambda}_n,\ n \in \mathbb{N}_0\}$ is a non-decreasing sequence which tends to infinity.

Denote

$$\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k), \quad k \in \mathbb{N}_0 ,$$

where $\mathcal{R}^0 = J$ and J is the identity operator.

Now, we shall define the scale of spaces $\mathcal{A}_k$, $k \in \mathbb{N}_0$. Our construction

$$\mathcal{A}_k = \left\{ \phi \in L^2(I):\ \phi = \sum_{n=0}^{\infty} a_n\psi_n,\ \sum_{n=0}^{\infty} |a_n|^2\, \tilde{\lambda}_n^{2k} < \infty \right\}, \quad k \in \mathbb{N}_0 .$$

We see that $\mathcal{A}_0 = L^2(I)$. The space $\mathcal{A}_k$ is the Hilbert space equipped with the scalar product

$$(\phi,\psi)_k = \sum_{n=0}^{\infty} a_n \bar{b}_n\, \tilde{\lambda}_n^{2k}, \qquad \phi,\psi \in \mathcal{A}_k$$

and the norm

$$\|\phi\|_k = \left(\sum_{n=0}^{\infty} |a_n|^2\, \tilde{\lambda}_n^{2k} \right)^{1/2}, \quad \phi \in \mathcal{A}_k ,$$

where $\phi = \sum_{n=0}^{\infty} a_n\psi_n$, $\psi = \sum_{n=0}^{\infty} b_n\psi_n$,

Note that the orthonormal system in $\mathcal{A}_k$ is $\tilde{\psi}_j = \dfrac{\psi_j}{\tilde{\lambda}_j^k}$, $j \in \mathbb{N}_0$.

2. Put

$$S = \left\{ \phi = \sum_{n=0}^{m} a_n\psi_n :\ m \in \mathbb{N}_0,\ \ a_n \in \mathbb{C} \right\}.$$

The set S is dense in $\mathcal{A}_k$, $k \in \mathbb{N}_0$. The operator $\mathcal{R}^m$, $m \in \mathbb{N}_0$ is defined on S. From the fact that the mapping $\mathcal{R}^m : S \to L^2(I)$ is linear and continuous, it follow that $\mathcal{R}^m$, $m \le k$ can be extended linearly and continuously to the space $\mathcal{A}_k$. Denote this extension by $\tilde{\mathcal{R}}^m$, $m \le k$. Let $\phi_p = \sum_{n=0}^{\infty} a_n\psi_n \in S$ and $\phi = \sum_{n=0}^{\infty} a_n\psi_n \in \mathcal{A}_k$. We have that $\phi_p \to \phi$, $p \to \infty$, in $\mathcal{A}_k$, so

$$\tilde{\mathcal{R}}^m\phi = \tilde{\mathcal{R}}^m \left(\sum_{n=0}^{\infty} a_n\varphi_n \right) = \lim_{p\to\infty} \left[\mathcal{R}^m \left(\sum_{n=0}^{p} a_n\psi_n \right) \right] = \sum_{n=0}^{\infty} a_n\, \lambda_n^m\, \psi_n .$$

Let $\phi \in \mathcal{A}_k \cap C^\infty(I)$ and $\langle \mathcal{R}^m\phi,\ \psi_n \rangle = \langle \phi,\ \mathcal{R}^m\psi_n \rangle$, $m \le k$, $n \in \mathbb{N}_0$, where $\langle \phi,\psi \rangle = = \int_I \phi(t)\ \psi(t)dt$, $\phi,\psi \in L^2(I)$. Then $\tilde{\mathcal{R}}^m = \mathcal{R}^m\phi$, $m \le k$.

Next, we shall define tha spaces $\mathcal{A}_{-k}$ $k \in \mathbb{N}_0$ in the following formal way:

$$\mathcal{A}_{-k} = \left\{ f:\ f = \sum_{n=0}^{\infty} b_n\psi_n,\ \sum_{n=0}^{\infty} |b_n|^2\, \tilde{\lambda}_n^{-2k} < \infty \right\}, \quad k \in \mathbb{N}_0 .$$

The set $\mathcal{A}_{-k}$ is a vector space (with operations defined in the usual way).

We can define a scalar product and a norm on it.

Namely, let $f = \sum_{n=0}^{\infty} b_n \psi_n$, $g = \sum_{n=0}^{\infty} c_n \psi_n$, then

$$(f,g)_{-k} = \sum_{n=0}^{\infty} b_n \bar{c}_n \tilde{\lambda}_n^{-2k}$$

$$\|f\|_{-k} = \left(\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} \right)^{1/2}$$

It is obvious that $\mathcal{A}_{-k}$ is a Hilbert space.

Let $\mathcal{A}'_k$ be the dual space of $\mathcal{A}_k$, $k \in \mathbb{N}_0$. We have

Theorem 2.1. *There is an isometry between spaces* $\mathcal{A}'_k$ *and* $\mathcal{A}_{-k}$.

Proof. Let $f \in \mathcal{A}'_k$. Denote by $b_n = (f, \psi_n) = f(\psi_n) = \langle f, \bar{\psi}_n \rangle$, $n \in \mathbb{N}_0$, and let $\phi = \sum_{n=0}^{\infty} a_n \psi_n \in \mathcal{A}_k$. Since f is linear and continuous, we have that

$$(f,\phi) = \sum_{n=0}^{\infty} \bar{a}_n b_n . \tag{2.1}$$

It follows from [6] that

$$\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} < \infty . \tag{2.2}$$

So, there exists an element $g \in \mathcal{A}_{-k}$ such that $g = \sum_{n=0}^{\infty} b_n \psi_n$. Conversely, if we have $g = \sum_{n=0}^{\infty} b_n \psi_n \in \mathcal{A}_{-k}$, such that relation (2.1) holds, then the mapping $\phi = \sum_{n=0}^{\infty} a_n \psi_n \to \sum_{n=0}^{\infty} \bar{a}_n b_n$, $\phi \in \mathcal{A}_{-k}$, defines an element from $\mathcal{A}'_k$. Denote this element by f. It is obvious that $b_n = (f, \psi_n)$, $n \in \mathbb{N}_0$. Hence, we have a one-to-one mapping $f \in \mathcal{A}'_k \to \sum_{n=0}^{\infty} b_n \psi_n \in \mathcal{A}_{-k}$, where $b_n = (f, \psi_n)$, $n \in \mathbb{N}_0$. Obviously, this mapping is linear.

Next, we shall prove that $|f|'_k = \|f\|_{-k}$, where $|f|'_k$ is the dual norm in $\mathcal{A}'_k$.

We have

$$|(f,\phi)| = \left| \sum_{n=0}^{\infty} b_n \bar{a}_n \right| \le \left(\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} \right)^{1/2} \left(\sum_{n=0}^{\infty} |a_n|^2 \tilde{\lambda}_n^{2k} \right)^{1/2}$$

$$|(f,\phi)| \le \|f\|_{-k} \|\phi\|_k$$

$$|f|' \le \|f\| .$$

Furthermore let $\phi_m = \sum_{n=0}^{m} b_n \tilde{\lambda}_n^{-2k} \psi_n \in \mathcal{A}_k$. We have

$$\|\phi_m\|_k = \left(\sum_{n=0}^{\infty} |b_n|^2 |\tilde{\lambda}_n^{-2k}|^2 \tilde{\lambda}_n^{2k} \right)^{1/2} = \left(\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} \right)^{1/2},$$

so that

$$|f|'_k = \frac{(f,\phi_m)}{\|\phi_m\|_k} = \frac{\sum_{n=0}^{m} |b_n|^2 \tilde{\lambda}_n^{-2k}}{\left(\sum_{n=0}^{m} |b_n|^2 \tilde{\lambda}_n^{-2k} \right)^{1/2}} = \left(\sum_{n=0}^{m} |b_n|^2 \tilde{\lambda}_n^{-2k} \right)^{1/2} \xrightarrow[m \to \infty]{} \left(\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} \right)^{1/2} = \|f\|_{-k}.$$

It follows that $|f|'_k = \|f\|_{-k}$.

We shall write $X \hookrightarrow Y$ to denote that a topological vector space X can be embedded linearly and continuously into a topological vector space Y. One can prove easily that

$$\ldots \mathcal{A}_{k+1} \hookrightarrow \mathcal{A}_k \hookrightarrow \ldots \hookrightarrow \mathcal{A}_1 \hookrightarrow \mathcal{A}_0 = L^2(I) \hookrightarrow \mathcal{A}_{-1} \ldots \mathcal{A}_{-k} \hookrightarrow \ldots$$

Let

$$\mathcal{A} = \bigcap_{k=0}^{\infty} \mathcal{A}_k = \left\{ \phi \in L^2(I): \quad \phi = \sum_{n=0}^{\infty} a_n \psi_n; \quad \forall k \sum_{n=0}^{\infty} |a_n|^2 \tilde{\lambda}_n^{2k} < \infty \right\}.$$

$$\mathcal{A} = \bigcap_{k=0}^{\infty} \mathcal{A}_{-k} = \left\{ f: \quad f = \sum_{n=0}^{\infty} b_n \psi_n, \quad \exists k \sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} < \infty \right\}.$$

Note that the space $\mathcal{A}$ is dense in $\mathcal{A}_k$, $k \in \mathbb{N}_0$, because it contains the set S which is dense in each $\mathcal{A}_k$, $k \in \mathbb{N}_0$. So, $\mathcal{A}_k$, $k \in \mathbb{N}_0$, is the completion of $\mathcal{A}$ with respect to the norm $\|\cdot\|_k$. From Theorem 2.1. it follows that $\mathcal{A}'$ is the dual of $\mathcal{A}$, 6. From 11, Lemma 9.3.3, p.316 it follows that the spaces $\mathcal{A}$ and $\mathcal{A}'$ are identical to the spaces defined in 11, ch.9.3. and 9.6. and denoted by the same letters.

Definition. *An element from* $\mathcal{A}_{-k}$ *(i.e.* $\mathcal{A}'_k$*) is called a generalized function of* $\mathcal{R}$*-order* k. *An element from* $\mathcal{A}'$ *is called a generalized function of* $\mathcal{R}$*-finite order.*

2.3. Let $m, k \in \mathbb{N}_0$. In Section 2.2. we defined the mappings $\mathcal{R}^m: \mathcal{A}_k \to L^2(I)$, $m \le k$. We define the mappings $(\mathcal{R}^m)': L^2(I) \to \mathcal{A}_{-k}$, $m \le k$, in the following way

(2.3) $((\mathcal{K}^m)'f,\phi) = (f,\mathcal{K}^m\phi),\ m \le k,\ \ \phi \in \mathcal{A}_k,\ \ f \in L^2(I)$.

If $f \in L^2(I)$ is the of form $f = \sum_{n=0}^{\infty} b_n\psi_n$, $\sum_{n=0}^{\infty} |b_n|^2 \le \infty$, we have

$$(2.4)\qquad (\mathcal{K}^m)'f = \sum_{n=0}^{\infty} b_n\, \lambda_n^m\, \psi_n .$$

It is obvious that $(\mathcal{K}^m)'$ can be defined on $\mathcal{A}_{-p}$, $p \in N$, in the same way as in (2.3) and that $(\mathcal{K}^m)' : \mathcal{A}_{-p} \to \mathcal{A}_{-p-m}$. As it formally $(\mathcal{K}^m)'f = \sum_{n=0}^{\infty} b_n\lambda_n^m\psi_n$, $f \in \mathcal{A}_{-k}$, $m \le k$, we shall denote $(\mathcal{K}^m)'$ by $\mathcal{K}^m$, $m \le k$,

Put $\Lambda = \{n \in \mathbb{N}_0 : \lambda_n = 0\}$, and $\Lambda^c = \mathbb{N}_0 \backslash \Lambda$. It is easy to prove the following representation theorem

Theorem 2.2. *Let* $f \in \mathcal{A}_{-k}$ *be of the form* $f = \sum_{n=0}^{\infty} b_n\psi_n$, *and* $F = \sum_{n\in\Lambda^c} (b_n\lambda_n^{-k})\psi_n$.

Then, we have that $F \in L^2(I)$ *and*

$$f = \mathcal{K}^k F + \sum_{n\in\Lambda} b_n\psi_n .$$

So

$$f \in \mathcal{A}' \ \Leftrightarrow \exists k \in \mathbb{N}_0,\ \exists F \in L^2(I),\ \ \exists b_n \in \mathbb{C},\ n \in \Lambda,\ f = \mathcal{K}^k F + \sum_{n\in\Lambda} b_n\psi_n .$$

3. Generalized random processes from $L(\mathcal{A}_k, Z)$, $k \in \mathbb{N}_0$.

3.1. Let $(\Omega,\mathcal{F},P)$ be a probability space. Denote by Z the space of all the P-equivalence classes of complex random variables with finite second momenta. Z is the Hilbert space with the scalar product and the norm defined in the usual way. For ξ, $\eta \in Z$,

$$(\xi,\eta)_Z = E\xi\bar{\eta} = \int_\Omega \xi(\omega)\bar{\eta}(\omega)dP(\omega) ,$$

$$\|\xi\|_Z = (\xi,\xi)_Z^{1/2} .$$

We suppose that Z is separable, so there exists o.n.s $\{\xi_n,\ n \in \mathbb{N}_0\}$ and foe $\xi \in Z$ we have $\xi = \sum_{}^{\infty} c_n\xi_n$, $c_n = (\xi,\xi_n)_Z$, $n \in \mathbb{N}_0$, $\sum_{}^{\infty} |c_n|^2 < \infty$.

In the definition and assertion which are to follow in Section 3. and 4. we follow [4].

Definition. *Random process* $\xi(t)$ *on* I *is a family* $\{\xi(t),\ t \in I\}$ *of random variables from* Z.

Denote by $L(X,Y)$ a vector space of all the linear and contonuous mappings from a topological vector space X to a topological space Y.

Definiton. *Generalized random process on* $\mathcal{A}$ *is an element from* $L(\mathcal{A},Z)$. *Denote* $\mathcal{A}^{\bullet} = L(\mathcal{A},Z)$, *and by* (ξ,ϕ) *the value of* $\xi \in \mathcal{A}^{\bullet}$ *at* $\phi \in \mathcal{A}$. *A sequence* $\{\xi_n, n \in \mathbb{N}\}$ *converges to* $\xi \in \mathcal{A}^{\bullet}$ *if for each* $\phi \in \mathcal{A}$, $\lim_{n\to\infty} (\xi_n,\phi) = (\xi,\phi)$ *in* Z.

We can define the operator $\mathcal{R}^k$ on the space $\mathcal{A}^{\bullet}$ in the following way

$$(\mathcal{R}^k \xi,\phi) = (\xi,\mathcal{R}^k \phi), \quad k \in \mathbb{N}_0 .$$

Denote $\mathcal{A}_k^{\bullet} = L(\mathcal{A}_k, Z)$. The norm on $\mathcal{A}_k^{\bullet}$ is defined in the following way

$$\|\eta\|^{\bullet}_{-k} = \sup\left\{\|(\eta,\phi)\|_Z, \quad \phi \in \mathcal{A}\ , \ \|\phi\| \le 1\right\} .$$

The space $\mathcal{A}_k^{\bullet}$ is completa because Z is complete. The relation $\mathcal{A}_k' = \mathcal{A}_{-k} \hookrightarrow \mathcal{A}_k^{\bullet}$ holds, and for $f \in \mathcal{A}_{-k}$ we have $\|f\|^{\bullet}_{-k} = \|f\|_{-k} = |f|_k'$. Since for $n \ge m \ge 0$ we have $\mathcal{A}_n \hookrightarrow \mathcal{A}_m$ and $\|\phi\|_m \le \|\phi\|_n$, it follows that $\mathcal{A}_m^{\bullet} \hookrightarrow \mathcal{A}_n^{\bullet}$. Also, $\mathcal{A} \hookrightarrow \mathcal{A}_k$, and every convergent sequence in $\mathcal{A}$ is convergent in $\mathcal{A}_k$, so $\mathcal{A}_k^{\bullet} \hookrightarrow \mathcal{A}^{\bullet}$. Therefore, the spaces $\mathcal{A}_k^{\bullet}$ satisfy:

$$(L^2(I))^{\bullet} = \mathcal{A}_0^{\bullet} \hookrightarrow \mathcal{A}_1^{\bullet} \ \dots\ \mathcal{A}_k^{\bullet} \hookrightarrow \dots \hookrightarrow \mathcal{A}^{\bullet}$$

and moreover

$$\mathcal{A}^{\bullet} = \bigcup_{k=0}^{\infty} \mathcal{A}_k^{\bullet}, \quad \text{(in the set theoretical sense)}$$

Definition. *An element from* $\mathcal{A}_k^{\bullet} = L(\mathcal{A}_k,Z)$ *is called the generalized random process of* $\mathcal{R}$*-order* k.

3.2. Denote by Y^{∞} a space of random processes on I of the form

$$\eta(t,\omega) = \sum_{n=0}^{m} a_n(\omega)\phi_n(t)\ , \quad a_n(\omega) \in Z, \quad t \in I .$$

Obviously $S \subset Y^{\infty}$.

Lemma 3.1. *Y^{∞} is a subspace of $\mathcal{A}_k^{\bullet}$,* $k \in \mathbb{N}_0$.

Proof. Let $\eta \in Y^{\infty}$ be of the form $\eta = \sum_{n=0}^{m} a_n(\omega)\psi_n$. For $\phi \in \mathcal{A}_k$, $\phi = \sum_{n=0}^{m} b_n\psi_n$ we have

$$(\eta,\phi) = \left[\sum_{n=0}^{m} a_n(\omega)\psi_n, \phi\right] = \sum_{n=0}^{m} a_n(\omega)(\psi_n,\phi) = \sum_{n=0}^{m} a_n(\omega)\bar{b}_n ,$$

so $(\eta,\phi) \in Z$, for every $\phi \in \mathcal{A}_k$. Linearly is obvious.

To check continuity, let $\phi_n, \phi \in \mathcal{A}_k$, $n \in \mathbb{N}$ be of the form $\phi_n = \sum_{l=1}^{\infty} b_l^n\psi_l$, $\phi = \sum_{l=0}^{\infty} b_l\psi_l$, and let ϕ_n converge to ϕ in $\mathcal{A}_k$, i.e.

$$\sum_{l=0}^{\infty} |b_l^n - b_l|^2\, \tilde{\lambda}_l^{2k} \to 0, \quad n \to \infty$$

Therefore,

$$(\eta,\phi_n - \phi) = \sum_{l=0}^{m} a_l(\omega)(b_l^n - b_l) \to 0, \quad n \to \infty$$

in Z.

Denote by Y^{-k} the space obtained by completing Y^{∞} in $\mathcal{A}_k^{\bullet}$ with respect to the norm $\|\cdot\|_{-k}^{\bullet}$,

Lemma 3.2. *$\mathcal{A}_{-k}$ is a subspace of Y^{-k}.*

Proof. Let $f \in \mathcal{A}_{-k}$ be of the form $f = \sum_{n=0}^{\infty} a_n\psi_n$ and $\eta_m \in Y^{\infty}$ of the form $\eta_m = \sum_{n=0}^{m} a_n\psi_n$, $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$. We shall show that the sequence η_m convergence to f in Y^{-k}.

$$\|f - \eta_m\|_{-k}^{\bullet} = |f - \eta_m|'_k = \|f - \eta_m\|_{-k} = \left[\sum_{n=m+1}^{\infty} |a_n|^2\tilde{\lambda}_n^{-2k}\right]^{1/2} \to 0, \quad m \to \infty$$

Theorem 3.1. *Y^{-k} is a proper subspace of $\mathcal{A}_k^{\bullet}$,* $k \in \mathbb{N}_0$.

Proof. Let $\{\xi_n,\ n \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_n,\ n \in \mathbb{N}_0\}$ be o.n.s. in Z and $\mathcal{A}_k$ respectively. Let $\phi = \sum_{n=0}^{\infty} c_n\psi_n = \sum_{n=0}^{\infty} c_n\tilde{\lambda}_m^k\tilde{\psi}_n$. Define an element $\eta^{\bullet} \in L(\mathcal{A}_k, Z)$ by

$$(\eta^{\bullet},\phi) = \sum_{n=0}^{\infty} (\tilde{\psi}_n\phi)_k\ \xi_n . \tag{3.1}$$

The mapping is well defined since

$$\|(\eta^\bullet, \phi)\|_Z^2 = \sum_{n=0}^{\infty} |(\tilde\psi,\phi)_k|^2 = \sum_{n=0}^{\infty} |\bar c_n|^2 \tilde\lambda_n^{2k} < \infty ,$$

where

$$(\tilde\psi,\phi)_k = \left(\frac{\psi_n}{\tilde\lambda_n^k}, \sum_{n=0}^\infty c_n\psi_n\right)_k = \frac{\bar c_n \tilde\lambda_n^{2k}}{\tilde\lambda_n^k} = \bar c_n \tilde\lambda_n^k .$$

Linearly is obvious and if $\phi_m \to \phi$ in $\mathcal{A}_k$, then

$$\|(\eta^\bullet, \phi_m - \phi)\|_Z^2 = \sum_{n=0}^\infty |c_n^m - c_n|^2 \tilde\lambda_n^{2k} \to 0, \quad m\to\infty ,$$

so that $\eta^\bullet$ is continuous and $\eta^\bullet \in \mathcal{A}_k^\bullet$.

The mappings $\phi \mapsto (\tilde\psi_n,\phi)$ $n\in\mathbb{N}_0$, are linear and continuous, with norms equal to 1, so there exist $f_n \in \mathcal{A}_{-k}$, $n\in\mathbb{N}_0$, (Theorem 2.1.) such that $\|f_n\|_{-k}=1$ and

$$(\tilde\psi_n,\phi) = (f_n,\phi) , \quad n\in\mathbb{N}_0,$$

and

$$(\eta^\bullet,\phi) = \sum_{n=0}^\infty (f_n,\phi)\xi_n .$$

Let η be an arbitrary element from Y^∞ of the form

$$\eta(t,\omega) = \sum_{n=0}^m d_n(\omega)\,\tilde\psi_n(t) .$$

For $t\in I$, fixed, we have

$$\eta(t,\omega) = \sum_{n=0}^m g_n(t)\,\xi_n(\omega) ,$$

where

$$g_n(t) = (\eta(t,\omega),\ \xi_n(\omega))_Z , \quad n\in\mathbb{N}_0.$$

We shall prove

$$\sum_{n=0}^\infty \int_I |g_n(t)|^2 dt < \infty . \tag{3.2}$$

We have that

$$\int_I \|\eta(t,\omega)\|_Z^2\, dt = \int_I \left[\int_\Omega \Big|\sum_{n=0}^m d_n(\omega)\,\tilde\psi_n(t)\Big|^2 dP(\omega)\right] dt \le$$

$$\le \int_I \left\{ \int_\Omega \left[\sum_{i,j\le m} d_i(\omega)\,\tilde\psi_i(t)\,\bar d_j(\omega)\,\bar{\tilde\psi}_j(t)\right] dP(\omega)\right\} dt =$$

$$= \sum_{i,\, j \le m} \int_I \tilde{\psi}_i(t)\, \bar{\tilde{\psi}}_j(t)dt \int_\Omega d_i(\omega)\bar{d}_j(\omega)dP(\omega) =$$

$$= \sum_{n=0}^{m} \frac{1}{\tilde{\lambda}_n^{2k}} \|d_n(\omega)\|_Z^2 < \infty .$$

Furthermore, since

$$\int_I \sum_{n=0}^{\infty} |g_n(t)|^2 dt = \int_I \|\eta(t,\omega)\|_Z^2 dt < \infty ,$$

and according Lebesgue's theorem

$$\sum_{n=0}^{\infty} \int_I |g_n(t)|^2 dt = \int_I \sum_{n=0}^{\infty} |g_n(t)|^2 dt < \infty ,$$

(3.1) follows.

Hence, $\|g_n(t)\|_0^2 = \int_I |g_n(t)|^2 dt \to 0$, $n \to \infty$. Since $\|f_n\|_{-k} = 1$ and $\|g_n\|_{-k} \le \|g_n\|_0 \to 0$, we have

$$\|\eta^\bullet - \eta\|_{-k}^{\bullet 2} = \sup \left\{ \|(\eta^\bullet - \eta, \phi)\|_Z^2 \ ,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1 \right\} =$$

$$= \sup \left\{ \sum_{n=0}^{\infty} |(f_n - g_n, \phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1 \right\} \ge$$

$$\ge \lim_{n\to\infty} \sup \left\{ |(f_n - g_n, \phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1 \right\} =$$

$$= \lim_{n\to\infty} \sup \left\{ |(f_n, \phi)|^2 - 2|(f_n, \phi)| \cdot |(g_n, \phi)| + |(g_n, \phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1 \right\} \ge$$

$$\ge \lim_{n\to\infty} \left\{ \|f_n\|_{-k}^2 - 2\|f_n\|_{-k} \|g_n\|_{-k} \right\} = 1.$$

Since Y^∞ is dense in Y^{-k}, we have that for any $\eta \in Y^{-k}$, $\|\eta^\bullet - \eta\|_{-k} \ge 1$. It follows that the element $\eta^\bullet$, defined in (3.1) does not belong to Y^{-k}, so Y^{-k} is the proper subspace of $\mathcal{A}_k^\bullet$.

4. Structural theorems

4.1. Definiton. *Let* $\{f_n,\ n \in \mathbb{N}_0\}$ *and* $\{\theta_n,\ n \in \mathbb{N}_0\}$ *be sequences from* $\mathcal{A}'$ *and* Z *respectively.Then* $\sum_{n=0}^{\infty} f_n \otimes \theta_n$ *is a generalized random proces in* $\mathcal{A}^\bullet$ *defined by*

$$\left[\sum_{n=0}^{\infty} f_n \otimes \theta_n, \phi \right] = \lim_{m\to\infty} \sum_{n=0}^{m} (f_n,\phi)\ \theta_n, \quad \forall\phi \in \mathcal{A},$$

provided that the limit on the right-hand side exists in Z *for each* $\phi \in \mathcal{A}$.

Linearity is obvious and continuity follows form the Banach Steinhaus theorem, so that our definition is correct.

Theorem 4.1. *Let* $\eta \in \mathcal{A}^\bullet$ *and* $\{\xi_n,\ n \in \mathbb{N}_0\}$ *be o.n.s in* Z. *Then* η *belongs to* $\mathcal{A}^\bullet_k$, $k \in \mathbb{N}_0$ *if and only if* η *is expressible in the form*

$$\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n, \tag{4.1}$$

where $f_n \in \mathcal{A}_{-k}$, $n \in \mathbb{N}_0$ and for every $\phi \in \mathcal{A}_k$

$$\sum_{n=0}^{\infty} |(f_n\phi)|^2 < \infty . \tag{4.2}$$

Proof. Let $\eta \in \mathcal{A}^\bullet_k$. Then the mappings $\phi \to ((\eta,\phi),\xi_n)_Z$ are in $\mathcal{A}'_k = \mathcal{A}_{-k}$ for every $\xi_n \in Z$, $n \in \mathbb{N}_0$. So, there exist $f_n \in \mathcal{A}_{-k}$, $n \in \mathbb{N}_0$ such that

$$((\eta,\phi),\xi_n)_Z = (f_n,\phi), \quad \forall\phi \in \mathcal{A}_k, \quad n \in \mathbb{N}_0 .$$

and

$$(\eta,\phi) = \sum_{n=0}^{\infty} ((\eta,\phi),\xi_n)_Z \xi_n = \left[\sum_{n=0}^{\infty} (f_n \otimes \xi_n),\phi \right], \quad \forall\phi \in \mathcal{A}_k .$$

So (4.1) follows.

Further, we have

$$\infty > \|\eta,\phi)\|_Z^2 = \sum_{n=0}^{\infty} |((\eta,\phi),\xi_n)_Z|^2 = \sum_{n=0}^{\infty} |(f_n,\phi)|^2, \quad \forall\phi \in \mathcal{A}_k ,$$

which proves (4.2).

Conversely, let $\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n$, $f_n \in \mathcal{A}_{-k}$, $\sum_{n=0}^{\infty} |(f_n,\phi)|^2 < \infty$, $\forall\phi \in \mathcal{A}_k$.

Since $\eta \in \mathcal{A}^\bullet$ we have

$$(\eta,\phi) = \sum_{n=0}^{\infty} (f_n,\phi)\xi_n \in Z, \quad \forall\phi \in \mathcal{A}.$$

Consider the sequence

$$\eta_m = \sum_{n=0}^{m} f_n \otimes \xi_n .$$

It is easy to prove that η_m is a Cauchy sequence in $\mathcal{A}_k^{\bullet}$. Namely, it is obvious that $\eta_m \in \mathcal{A}_k^{\bullet}$, $m \in \mathbb{N}_0$, and for $\phi \in \mathcal{A}_k$ and $l > k$.

$$\|(\eta_l,\phi) - (\eta_k,\phi)\|_Z^2 = \sum_{n=k+1}^{l} |(f_n,\phi)|^2 < \varepsilon, \quad l,k > k_0(\varepsilon).$$

Since $\mathcal{A}_k^{\bullet}$ is complete, the sequence η_m converges in $\mathcal{A}_k^{\bullet}$ to an element $\eta_0 \in \mathcal{A}_k^{\bullet}$.

Let $\eta_0 = \sum_{n=0}^{m} \tilde{f}_n \otimes \xi_n$. We shall show that $\eta_0 = \eta$.

Since $\eta \to \eta_0$ in $\mathcal{A}_k^{\bullet}$, for every $n \in \mathbb{N}_0$, we have

$$0 = \lim_{m\to\infty} ((\eta_0 - \eta_m,\phi),\xi_n)_Z = \lim_{m\to\infty} \Big[((\eta_0,\phi),\xi_n)_Z - ((\eta_m,\phi),\xi_n)_Z\Big] =$$
$$= ((\eta_0,\phi),\xi_n)_Z - ((\eta,\phi),\xi_n)_Z = (\tilde{f}_n,\phi) - (f_n\phi).$$

So we have

$$((\tilde{f}_n,\phi) = (f_n,\phi), \quad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \forall\phi \in \mathcal{A}_k.$$

It follows that $\eta_0 = \eta$, $\eta_m \to \eta$ in $\mathcal{A}_k^{\bullet}$ and η is of the form $\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n$.

Theorem 4.2. *Let* $\eta \in \mathcal{A}^{\bullet}$ *and* $k \in \mathbb{N}_0$. *Then* η *belongs to* Y^{-k} *if and only if it can be represented in the form*

$$\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n . \tag{4.3}$$

where $f_n \in \mathcal{A}_{-k}$ *and for every* $\phi \in \mathcal{A}_k$, $\sum_{n=0}^{\infty} |(f_n,\phi)|^2 < \infty$, *and the sequence*

$$\eta_0 = \sum_{n=0}^{m} f_n \otimes \xi_n . \tag{4.4}$$

is a Cauchy sequence in Y^{-k}.

Proof. Let η be in Y^{-k}. From Theorem 3.1. $Y^{-k} \subset \mathcal{A}_k^{\bullet}$, so according to Theorem 4.1. $\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n$ where $f_n \in \mathcal{A}_{-k}$ and $\sum_{n=0}^{\infty} |(f_n,\phi)|^2 < \infty$ for every $\phi \in \mathcal{A}_k$. To prove that $\{\eta_m,\ m \in \mathbb{N}_0\}$ is the Cauchy sequence in Y^{-k} we shall show that $\eta_m \in Y^{-k}$, $m \in \mathbb{N}_0$, first. Since the set S is dense in $\mathcal{A}_{-k}$, it follows that for every $f_n \in \mathcal{A}_{-k}$, $n \in \mathbb{N}_0$, there exists a sequence $\{f_n^i,\ i \in \mathbb{N}_0\}$ in S such that $\|f_n^i - f_n\|_{-k} \to 0$, $i \to \infty$.

Define

$$\eta_m^i = \sum_{n=0}^{m} f_n^i\, \xi_n\,, \quad i \in \mathbb{N}_0\,;\quad m \in \mathbb{N}_0.$$

For every $i, m \in \mathbb{N}_0$, η_m^i is in Y^{∞} because f_n^i, $n, i \in \mathbb{N}_0$ are in S.

Furthermore,

$$\|\eta_m - \eta_m^i\|_{-k}^{*2} = \Big\|\sum_{n=0}^{m} (f_n - f_n^i)\xi_n\Big\|_{-k}^{*2} = \sum_{n=0}^{m} \|f_n - f_n^i\|_{-k}^2 \to 0, \quad i \to \infty.$$

So, for every $m \in \mathbb{N}_0$, η_m is in Y^{-k}.

Next, we shall prove that $\{\eta_m,\ m \in \mathbb{N}_0\}$ is a Cauchy sequence in Y^{-k}. Since η is in Y^{-k}, there exists a sequence $\{\theta_j,\ j \in \mathbb{N}_0\}$ in Y^{-k} such that $\|\eta - \theta_j\|_{-k}^{*} \to 0$, $j \to \infty$. Each θ_j can be represented in the form $\theta_j = \sum_{n=0}^{\infty} f_n^j\, \xi_n$ where $f_n^j = (\theta_j, \xi_n)_Z$ are all in S. Since $\{\xi_n,\ n \in \mathbb{N}_0\}$ is the complete o.n.s. in Z and θ_j is in Y^{∞} we have

$$\int_I \|\theta_j\|_Z^2\, dt = \sum_{n=0}^{\infty} \|f_n^j\|_0^2. \tag{4.5}$$

For arbitrary m and j there holds

$$\|\eta - \eta_m\|_{-k}^{*} \le \|\eta - \theta_j\|_{-k}^{*} + \|\theta_j - \eta_m\|_{-k}^{*}.$$

We have

$$\|\theta_j - \eta\|_{-k}^{*2} = \sup\Big\{\sum_{n=0}^{\infty} |(f_n^j - f_n, \phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\Big\}, \tag{4.6}$$

$$\|\theta_j - \eta_m\|_{-k}^{*2} = \sup\Big\{\sum_{n=0}^{m} |(f_n^j - f_n, \phi)|^2 + \sum_{n=m+1}^{\infty} |(f_n^j, \phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\Big\}, \tag{4.7}$$

$$\le \sup\Big\{\sum_{n=0}^{m} |(f_n^j - f_n, \phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\Big\} +$$

$$+ \sup\Big\{\sum_{n=m+1}^{\infty} |(f_n^j, \phi)|^2,\ \phi \in L^2(I),\ \|\phi\|_0 \le 1\Big\} \le$$

$$\le \sup\left\{ \sum_{n=0}^{\infty} |(f_n^j - f_n, \phi)|^2, \ \phi \in \mathcal{A}_k, \ \|\phi\|_k \le 1 \right\} + \sum_{n=m+1}^{\infty} \|f_n^j\|_0^2 =$$

$$= \|\theta_j - \eta\|_{-k}^{\bullet 2} + \sum_{n=m+1}^{\infty} \|f_n^j\|_0^2.$$

So, from (4.6) and (4.7) it follows that

$$\|\eta - \eta_m\|_{-k}^{\bullet} \le \|\eta - \theta_j\|_{-k}^{\bullet} + \left\{ \|\eta - \theta_j\|_{-k}^{\bullet 2} + \sum_{n=m+1}^{\infty} \|f_n^j\|_0^2 \right\}^{1/2}$$

Since $\theta_j \to \eta$ in Y^{-k}, for arbitrary $\varepsilon > 0$ there exist $j_0 = j_0(\varepsilon)$ such that $\|\eta - \theta_j\|_{-k}^{\bullet} \le \varepsilon/3$ for every $j \ge j_0$.

Furthermore, from (4.5) it follows that there exists $m_0 = m_0(j)$ such that $\sum_{n=m+1} \|f_n^j\|_0^2 < 3\varepsilon^2/9$, for all $m \ge m_0$. Hence, for every $j \ge j_0$ and $m \ge m_0$;

$$\|\eta - \eta_m\|_{-k}^{\bullet} \le \frac{\varepsilon}{3} + \left\{ \frac{\varepsilon^2}{9} + \frac{3\varepsilon^2}{9} \right\}^{1/2} = \varepsilon .$$

Since ε was arbitrary, it follows that the sequence $\{\eta_m, \ m \in \mathbb{N}_0\}$ converges to η in Y^{-k}, which means that it is the Cauchy sequence.

Conversly, let η and η_m be defined as in (4.3), (4.4). We shall show that η is in Y^{-k}. Since $\{\eta_m, \ m \in \mathbb{N}_0\}$ is a Cauchy sequence in Y^{-k}, it converges to an element; denote it by η_0, in Y^{-k}. We have that

$$\eta_0 = \sum_{n=0}^{\infty} f_n^0 \otimes \xi_n, \qquad \eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n \qquad \text{and} \qquad \eta_m = \sum_{n=0}^{m} f_n \otimes \xi_n ,$$

so,

$$0 = \lim_{m\to\infty} ((\eta_0 - \eta, \phi), \xi_n)_Z = \lim_{m\to\infty} \Big[((\eta_0 - \phi), \xi_n)_Z - ((\eta_m - \phi), \xi_n)_Z \Big] =$$

$$= ((\eta_0 - \phi), \xi_n)_Z - ((\eta_m - \phi), \xi_n)_Z = (f_n^0, \phi) - (f_n, \phi), \quad \forall n \in \mathbb{N}_0, \quad f \in \mathcal{A}_k.$$

We have $(f_n^0, \phi) = (f_n, \phi)$, $\forall n \in \mathbb{N}_0$, $\forall \phi \in \mathcal{A}_k$, i.e. $\eta = \eta_0$, which means η is in Y^{-k}.

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Rezime

JEDNA KLASA UOPŠTENIH SLUČAJNIH PROCESA SA VREDNOSTIMA U $L^2(\Omega)$

Date su strukturne teoreme za uopštene slučajne procese koji pripadaju prostorima $L(A_k, Z)$, Y^{-k}.

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CONVERGENCE OF A SEQUENCE OF GENERALIZED RANDOM PROCESSES ON THE ZEMANIAN SPACE $\mathcal{A}$

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Abstract

Different types of convergences of a sequence of generalized random processes on the Zemanian space $\mathcal{A}$ are defined and compared.

1. Introduction

Generalized random processes (g.r.p.) were defined by several authors [1,2,4,5,7,8,9,10,11,12]. Different types of convergences of a sequence of g.r.p.-s were introduced and investigated in [4]. In [1,2,4,5,7,10,11,12] spaces $\mathcal{D}$ and $K\{M_p\}$ were taken to be the spaces of test functions and in [8,9] the Zemanian space $\mathcal{A}$. In [1,4,5,7,8,9,10,12] the representation theorems for a g.r.p. were obtained.

For a space of test functions we take the space $\mathcal{A}$, whose elements have an orthogonal expansion. The space $\mathcal{A}$ and its dual space $\mathcal{A}'$ were introduced in [13]. Our construction of the spaces $\mathcal{A}$ and $\mathcal{A}'$ is different from [13], and the details are given in [8].

In [4] the representation theorems for a sequence of g.r.p. on $K\{M_p\}$ converging almost surely, in probability and mean (K') were obtained. Following [4], in [9] representation theorems for a sequence of g.r.p.-s on converging almost surely $(\mathcal{A}')$ are obtained.

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In Sections 2., 3. we shall give the basic definitions and properties of space $\mathcal{A}$ and of a g.r.p. on $\mathcal{A}$. In Section 4. we shall define various types of convergences of a sequence of g.r.p.-s and give representation theorems for sequence of g.r.p.-s converging in probability and mean ($\mathcal{A}'$).

2. Spaces $\mathcal{A}$ and $\mathcal{A}'$

We shall use the notation from [13]. Let I be an open interval of the real line $\mathbb{R}$ and $L^2(I)$ be the spaces of the equivalence classes of square integrable functions with values in the set of complex numbers, $\mathbb{C}$, with the usual norm. Denote by $C^\infty(I)$ the set of infinitely differentiable (smooth) functions, by $\mathbb{N}$ the set $\{1,2,\dots\}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{R}$ be a linear differential self-adjoint operator of the form

$$\mathcal{R} = \theta_0 D^{n_1} \theta_1 \; \dots \; D^{n_\nu} \theta_\nu,$$

such that

$$\mathcal{R} = \bar{\theta}_\nu (-D)^{n_\nu} \; \dots \; (-D)^{n_2} \bar{\theta}_1 (-D)^{n_1} \; \bar{\theta}_0 \; ,$$

where $D=d/dx$, $n_k \in \mathbb{N}_0$, $k=1,2,\dots,\nu$, θ_k, $k=0,1,\dots,\nu$, are smooth functions without zeros on I, and $\bar{\theta}_k$ are complex conjugates of θ_k, $k=0,1,\dots,\nu$, . We suppose that there exist a sequence of real numbers $\{\lambda_n,\ n \in \mathbb{N}_0\}$, and a sequence of smooth functions $\{\psi_n,\ n \in \mathbb{N}_0\}$ such that $\mathcal{R}\psi_n = \lambda_n \psi_n$, $n \in \mathbb{N}_0$. Furthermore, suppose that the sequence $\{\lambda_n,\ n \in \mathbb{N}_0\}$ monotonically tends to infinity and that $\{\psi_n,\ n \in \mathbb{N}_0\}$ forms a complete orthonormal system in $L^2(I)$. We can enumerate the sequences $\{\lambda_n,\ n \in \mathbb{N}_0\}$ and $\{\psi_n,\ n \in \mathbb{N}_0\}$, so that $|\lambda_0| \le |\lambda_1| \le |\lambda_2| \le \dots$. Put $\tilde{\lambda}_n = \lambda_n$ if $\lambda_n \neq 0$ and $\tilde{\lambda}_n = 1$, if $\lambda_n = 0$, $n \in \mathbb{N}_0$. The sequence $\{\tilde{\lambda}_n,\ n \in \mathbb{N}_0\}$ is nondecreasing and $|\tilde{\lambda}_n| \to \infty$. Let $\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k)$, $k \in \mathbb{N}_0$, where $\mathcal{R}^0 = \mathcal{J}$, $\mathcal{J}$ is the identity operator. In [8], the scale of spaces $\mathcal{A}_k$, $k \in \mathbb{N}_0$ is defined in the following way:

$$\mathcal{A}_k = \left\{ \phi \in L^2(I):\ \phi = \sum_{m=0}^{\infty} a_m \psi_m,\ \|\phi\|_k = \sum_{m=0}^{\infty} |a_m|^2 |\tilde{\lambda}_m|^{2k} < \infty \right\}, \quad k \in \mathbb{N}_0$$

Put

$$\mathcal{A} = \bigcap_{k=0}^{\infty} \mathcal{A}_k = \left\{ \phi \in L^2(I):\ f = \sum_{m=0}^{\infty} a_m \psi_m,\ \ \forall k,\ \|\phi\|_k < \infty \right\}.$$

The set

$$S = \left\{ \phi = \sum_{m=0}^{n} (a_m + ib_m)\, \psi_m,\ s \in \mathbb{N}_0,\ a_m, b_m \in Q \right\}$$

(Q is the set of rational numbers), is a countable dense set in each $\mathcal{A}_k$, $k \in \mathbb{N}_0$, and hence in $\mathcal{A}$. Also, since $S \subset \mathcal{A}$, $\mathcal{A}$ is dense in each $\mathcal{A}_k$, $k \in \mathbb{N}_0$. Thus $\mathcal{A}_k$, $k \in \mathbb{N}_0$, is the completition of $\mathcal{A}$ with respect to the norm $\|\cdot\|_k$.

Let $\mathcal{A}'$ ($\mathcal{A}'_k$) be the dual space of the space $\mathcal{A}$, ($\mathcal{A}'_k$) $k \in \mathbb{N}_0$. Then we have

$$\mathcal{A}' = \bigcup_{k=0}^{\infty} \mathcal{A}_k$$

From [13, ch. 9.3. and 9.6.] it follows that

$$(\psi_m,\ \mathcal{R}^k\phi) = (\mathcal{R}^k\psi_m, \phi), \quad m,k \in \mathbb{N}_0, \quad \phi \in \mathcal{A}$$

where for

$$\phi \in \mathcal{A}, \quad f \in \mathcal{A}', \quad (f,\phi) = \langle f, \bar{\phi} \rangle .$$

3. Generalized random processes on $\mathcal{A}$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. Throughout this paper we shall assume that $(\Omega, \mathcal{F}, \mathcal{P})$ is fixed.

Definition 3.1. *A generalized random processes on $\mathcal{A}$ is a mapping $\xi: \Omega \times \mathcal{A} \to \mathbb{C}$ such that*

(i) $\forall \phi \in \mathcal{A}$, $\xi(\cdot, \phi)$ *is a random variable on* Ω,

(ii) $\forall \omega \in \Omega$, $\xi(\omega, \cdot)$ *is an element from* $\mathcal{A}'$.

In [9] representation theorems for a g.r.p. on $\mathcal{A}$ were obtained. In this paper we shall need only the representation of a g.r.p. on $\mathcal{A}$ on a set $B \in \mathcal{F}$ with arbitrary large probability.

Theorem 3.1. *Let ξ be a g.r.p. on $\mathcal{A}$. Then for every $\varepsilon > 0$ there exist a set $B \in \mathcal{F}$, with $P(B) \geq 1-\varepsilon$, an integer $k_0 = k_0(\varepsilon) \in \mathbb{N}_0$, and a sequence of random variables on Ω, $\{c_m,\ m \in \mathbb{N}_0\}$ such that*

$$\xi(\omega, \phi) = \sum_{m=0}^{\infty} c_m(\omega)(\psi_m, \phi), \quad \omega \in B, \quad \phi \in \mathcal{A} \tag{3.1}$$

and

$$(3.2) \qquad \left[\sum_{m=0}^{\infty} |c_m(\omega)|^2 \, \tilde{\lambda}_m^{-2k_0} \right]^{1/2} < k_0 \, , \qquad \omega \in B \, .$$

The proof is given in [9, Theorem 3.1.]. See also [1,11,13] .

We define the differential operator $(\mathcal{R}')^k$, $k \in \mathbb{N}_0$ on the set of g.r.p.-s by

$$(\mathcal{R}')^k \xi(\omega,\phi) = \xi \, (\omega, \mathcal{R}^k\phi), \quad \omega \in \Omega, \quad \phi \in \mathcal{A} \, .$$

$$(\mathcal{R}')^{k+1} = \mathcal{R}'((\mathcal{R}')^k), \quad k \in \mathbb{N}_0 \qquad (\mathcal{R}')^0 = \mathcal{J}$$

We shall denote $\mathcal{R}'$ by $\mathcal{R}$. Put $\Lambda = \{ n \in \mathbb{N}_0 : \lambda_n = 0 \}$, $\Lambda^c = \mathbb{N}_0 \setminus \Lambda$.

Theorem 3.2. *Let ξ be a g.r.p. on $\mathcal{A}$. For every $\varepsilon > 0$ there exist a set $B \in \mathcal{F}$ with $P(B) \geq 1-\varepsilon$, an integer $k_0 = k_0(\varepsilon)$, a function $X_k : \Omega \times I \to \mathbb{C}$ and random variables $\{ c_m , \, m \in \mathbb{N}_0 \}$ such that for every $k \geq k_0$.*

$$(3.3) \qquad \xi(\omega,\phi) = \int_I X_k(\omega,t) \, \mathcal{R}^k\phi(t)dt + \sum_{m \in \Lambda} c_m(\omega)(\psi_m, \phi), \quad \omega \in B, \quad \phi \in \mathcal{A},$$

$$(3.4) \qquad \| X_k(\omega,\cdot) \|_{L^2} < k \, , \quad \omega \in B \, .$$

The proof of (3.3) is the same as in [9, Theorem 3.3. and 3.4.]. We note only that here, we shall take X_k in the form

$$X_k(\omega,t) = \sum_{m=0}^{\infty} b_m(\omega)\psi_m(t) \; , \qquad t \in I, \quad \omega \in \Omega \; ,$$

where

$$b_m(\omega) = \begin{cases} c_m(\omega) \, \tilde{\lambda}_m^{-k} \, , & \omega \in B \\ 0 \, , & \omega \notin B \end{cases} \qquad m \in \mathbb{N}_0 \, .$$

We have that $\| X_k(\omega,\cdot) \|_{L^2}$, $\omega \in \Omega$ is a random variable, since

$$\| X_k(\omega,\cdot) \|_{L^2} = \begin{cases} \sup \{ |\xi(\omega,\phi)|, \phi \in S_r, \; \|\phi\|_{k \leq 1} \}, & \omega \in B \\ 0 & , \; \omega \notin B \end{cases} =$$

$$= \begin{cases} \left[\sum_{m=0}^{\infty} |c_m(\omega)|^2 \, \tilde{\lambda}_m^{-2k} \right]^{1/2}, & \omega \in B \\ 0 & , \; \omega \notin B \end{cases} < k \, .$$

In [9] the following conditions were posed on sequence $\{\lambda_m,\ n \in \mathbb{N}_0\}$ and $\{\psi_m,\ m \in \mathbb{N}_0\}$ in order to obtain the representation with a continuous process X_k. By a continuous process on $\Omega \times I$ we shall mean the process that is, for almost every $\omega \in \Omega$, a continuous function on I.

(•) There exist $s_0 \in \mathbb{N}_0$ and a constant K such that, for $s \geq s_0$

$$\sup \left\{ |\psi_m(t)/\tilde{\lambda}_m^s| \ : \ m \in \mathbb{N}_0,\ \ t \in I \right\} < K .$$

(••) There exist $p_0 \in \mathbb{N}_0$ such that for $p \geq p_0$

$$\sum_{m=0}^{\infty} |\tilde{\lambda}_m|^{-2p} < \infty .$$

Theorem 3.3. *Let ξ be a g.r.p. on $\mathcal{A}$. Then, for every $\varepsilon > 0$ there exist a set $B \in \mathcal{F}$, with $P(B) \geq 1-\varepsilon$, an integer $k_0 = k_0(\varepsilon)$, random variables c_m, $m \in \Lambda$, and a continuous random process $X_k(\omega, t)$ on $\Omega \times I$, such that for $k \geq k_0$, $p \geq p_0$, $s \geq s_0$.*

$$(3.5) \qquad \xi(\omega,\phi) = \int_I X_k(\omega,t)\, \mathcal{R}^{k+p+s}\phi(t)dt + \sum_{m\in\Lambda} c_m(\omega)(\psi_m,\phi), \quad \omega \in B, \quad \phi \in \mathcal{A},$$

$$(3.6) \qquad \|X_k(\omega,\cdot)\|_{L^2} < k\ , \quad \omega \in \Omega\ .$$

The proof is similar to the proof of Theorem 3.5., [9], where the same representation as in (3.5) was obtained on a set $A \in \mathcal{F}$, with $P(A)=0$, under an additional condition. Relation (3.6) follows in the same way as in Theorem 3.2. . Again, we note that X_k has the form

$$X_k(\omega,t) = \begin{cases} \sum\limits_{m=0}^{\infty} c_m(\omega)\, \tilde{\lambda}_m^{-(k+p+s)} \psi_m(t)\ , & \omega \in B, \quad t \in I \\ 0 & ,\ \ \omega \notin B, \quad t \in I\ . \end{cases}$$

4. Convergence of generalized random processes on $\mathcal{A}$

We shall give the definitons of different types of convergences of a sequences of g.r.p.-s on $\mathcal{A}$, following [4].

Definiton 4.1. *The sequence $\{\xi_n,\ n \in \mathbb{N}_0\} = \{\xi_n\}$ of g.r.p.-s on $\mathcal{A}$ is said to converge to the g.r.p. ξ in probability ($\mathcal{A}'$) if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}_0$ such that*

$$\lim_{n\to\infty} P\left\{\omega\in\Omega \,\middle|\, \sup_{\|\phi\|_k\le 1} |\xi_n(\omega_1,\phi)-\xi(\omega,\phi)| \ge \varepsilon\right\} = 0$$

In short, we shall write

$$\xi_n \xrightarrow{P} \xi \ (\mathcal{A}')$$

Definition 4.2. *The sequence* $\{\xi_n\}$ *of g.r.p.-s on* $\mathcal{A}$ *is said to converge to the g.r.p.* ξ *im mean* $(\mathcal{A}')$ *if there exists* $k\in\mathbb{N}_0$ *such that*

$$\lim_{n\to\infty} \int_\Omega \sup_{\|\phi\|_k\le 1} |\xi_n(\omega,\phi)-\xi(\omega,\phi)|\,dP(\omega) = 0$$

In short, we shall write

$$\xi_n \xrightarrow{1} \xi \ (\mathcal{A}')$$

Obviously, convergences in probability [mean] $(\mathcal{A}')$ given above imply the weak convergences in probability [mean].

Definition 4.3. *(see also* [9]*). The sequence* $\{\xi_n\}$ *of g.r.p.-s on* $\mathcal{A}$ *is said to converge to a g.r.p.* ξ *almost surely* $(\mathcal{A}')$*, if there exists a set* $Z\in\mathcal{F}$*, with* $P(Z)=0$ *and for* $\omega\in\Omega\backslash Z$, $\xi_n(\omega,\cdot)\to\xi(\omega,\cdot)$ *weakly.*

In [9] representation theorems for a sequence of g.r.p.-s on $\mathcal{A}$ converging almost surely $(\mathcal{A}')$ were obtained. To obtain representation theorems of g.r.p.-s converging in probability and mean $(\mathcal{A}')$ we need a bound condition as in B(*ii*) of Theorem 4.1. of [9]. See also [4]. Thus we give:

Definition 4.4. *The sequence* $\{\xi_n\}$ *of g.r.p.-s on* $\mathcal{A}$ *is said to converge to the g.r.p.* ξ *boundedly in probability [mean]* $(\mathcal{A}')$*, if*

(*i*) $\xi_n \xrightarrow{P} \xi\ (\mathcal{A}')$, $[\xi_n \xrightarrow{1} \xi\ (\mathcal{A}')]$

(*ii*) *there exists a set* $Z\in\mathcal{F}$*, such that* $P(Z)=0$ *and for* $\omega\in\Omega\backslash Z$ $\{\xi_n(\omega,\cdot)\}$ *is bounded in* $(\mathcal{A}')$.

In short, we shall write $\xi_n \xrightarrow[b]{P} \xi\ (\mathcal{A}')$, $[\xi_n \xrightarrow[b]{1} \xi\ (\mathcal{A}')]$.

Obviously, $\xi_n \xrightarrow[b]{P} \xi\ (\mathcal{A}') \Rightarrow \xi_n \xrightarrow{P} \xi\ (\mathcal{A}')$ $[\xi_n \xrightarrow[b]{1} \xi\ (\mathcal{A}')] \Rightarrow [\xi_n \xrightarrow{1} \xi\ (\mathcal{A}')]$.

We have that (see [4,9]) condition (*ii*) of the above definition is equivalent to *(ii′) For every* $\varepsilon > 0$ *there exists set* $B \in \mathcal{F}$, *with* $P(B) \geq 1-\varepsilon$, *an integer* $k \in \mathbb{N}_0$, *independent of* n, *such that for every* $\omega \in B$, $\phi \in \mathcal{A}$ $|\xi_n(\omega,\phi)| \leq k\|\phi\|_k$.

Since $\xi_n \to \xi$ iff $\xi_n - \xi \to 0$, we shall consider the case $\xi_n \to 0$.

Theorem 4.1. *Let* $\{\xi_n\}$ *be a sequence of g.r.p.-s on* $\mathcal{A}$. *If* $\xi_n \to 0$ *boundedly in probability [mean]* $(\mathcal{A}')$, *then for every* $\varepsilon > 0$ *there exist a set* $B \in \mathcal{F}$, *such that* $P(B) \geq 1-\varepsilon$, *an integer* $k_0 \in \mathbb{N}_0$, *independent of* n, *and for every* $n \in \mathbb{N}_0$, *a sequence* $\{c_{m,n},\ m \in \mathbb{N}_0\}$ *of random variables on* Ω, *such that*

$$(4.1) \qquad \xi_n(\omega,\phi) = \sum_{m=0}^{\infty} c_{m,n}(\omega)(\psi_m,\phi), \quad \omega \in B, \quad \phi \in \mathcal{A}$$

$$(4.2) \qquad \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\right]^{1/2} < k_0, \quad \omega \in B.$$

(4.3) *for each* $\delta > 0$

$$P\left\{\omega \in B : \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\right]^{1/2} > \delta\right\} \to 0, \quad n \to \infty,$$

$$\left[\int_B \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\right]^{1/2} dP(\omega) \to 0, \quad n \to \infty\right].$$

$$(4.4) \qquad P\left\{\omega \in B : |c_{m,n}(\omega)| > \delta\right\} \to 0, \quad n \to \infty, \quad m \in \mathbb{N}_0$$

;

Proof. Assume that $\xi_n \xrightarrow[b]{P} 0$ $(\mathcal{A}')$, $[\xi_n \xrightarrow[b]{1} 0$ $(\mathcal{A}')]$ and let $\varepsilon > 0$ given. From equivalence (*ii*) and (*ii′*), there exist a set $B \in \mathcal{F}$, with $P(B) \geq 1-\varepsilon$ and an integer $k_0 \in \mathbb{N}_0$ such that for each $\omega \in B$ and $\phi \in \mathcal{A}$, $|\xi_n(\omega,\phi)| \leq k_0\|\phi\|_{k_0}$. Thus, (4.1) and (4.3) follow from Theorem 3.1.

We have that (see the proof of Theorem 3.1. of [9])

$$\sup_{\|\phi\|_{k_0} \leq 1} |\xi_n(\omega,\phi)| = \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\right]^{1/2}, \quad \omega \in B$$

Thus, for $\delta > 0$

$$P\left\{\omega \in B : \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\right]^{1/2} \geq \delta\right\} =$$

$$P\left\{\omega \in B : \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| \ge \delta\right\} \le$$

$$\le P\left\{\omega \in \omega : \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| \ge \delta\right\} \to 0, \quad n \to \infty .$$

$$\left[\int_B \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0}\right]^{1/2} dP(\omega) = \int_B \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| dP(\omega) \le\right.$$

$$\left.\le \int_\Omega \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| \, dP(\omega) \to 0 , \quad n \to \infty \right] .$$

Putting $\phi = \psi_m$ in (4.1) we get that, for $m \in \mathbb{N}_0$, $\delta > 0$

$$P\left\{\omega \in B : |c_{m,n}(\omega)| > \delta\right\} \to 0 , \quad n \to \infty .$$

To prove the converse, we need an additional assumption.

Theorem 4.2. *The sequence* $\{\xi_n\}$ *converges to zero boundedly in probability [mean]* ($\mathcal{A}'$), *if the following conditions hold.*

There exist $k \in \mathbb{N}_0$ *such that for every* $p \in \mathbb{N}$ *there exists a set* $B_p \in \mathcal{F}$, *with* $P(B) \ge 1 - \frac{1}{p}$, *such that*

(4.1') $$\xi_n(\omega,\phi) = \sum_{m=0}^{\infty} c_{m,n}(\omega)(\psi_m,\phi), \quad \omega \in B_p, \quad \phi \in \mathcal{A}$$

(4.2') $$\left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k}\right]^{1/2} < k , \quad \omega \in B_p ,$$

(4.3') *for every* $\delta > 0$

$$P\left\{\omega \in B_p : \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k}\right]^{1/2} > \delta\right\} \longrightarrow 0, \quad n \to \infty ,$$

$$\left[\int_{B_p} \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 \tilde{\lambda}_m^{-2k}\right]^{1/2} dP(\omega) \longrightarrow 0, \quad n \to \infty \right] .$$

Proof., Put $\varepsilon = 1/p$. Then we have that for every $p \in \mathbb{N}$ there exists a set $B_p \in \mathcal{F}$ with $P(B_p) \ge 1 - 1/p$, such that (4.1'), (4.2'), (4.3') hold. Let $\Omega_1 = \bigcup_{p=0}^{\infty} B_p$. We have that $P(\Omega_1) = 1$, and for $\omega \in \Omega$, $\phi \in \mathcal{A}$

$$|\xi_n(\omega,\phi)| = \Big[\sum_{m=0}^{\infty} c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \Big] \|\phi\|_k \ , \ \le k\|\phi\|_k ,$$

thus (*ii*) is satisfied.

To prove (*i*), let $\varepsilon > 0$ be given. Then, there exists, $p \in \mathbb{N}$ such that $P(B_p) \ge 1 - \frac{\varepsilon}{2}$. Also, from (4.3') it follows that there exists $n_0 = n_0(\varepsilon,\delta)$, such that for $n \ge n_0$, $\delta > 0$

$$P \Big\{ \omega \in B_p \ : \ \Big[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \Big]^{1/2} > \delta \Big\} < \frac{\varepsilon}{2}$$

Since

$$\sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| = \Big[\sum_{m=0}^{\infty} c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \Big]^{1/2} , \quad \omega \in B_p$$

we have for every $\delta > 0$, and $n \ge n_0$,

$$P\Big\{\omega \in \Omega_1 : \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| > \delta\Big\} = P\Big\{\omega \in B_p \ : \Big[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \Big]^{1/2} > \delta\Big\} +$$

$$+ \ P \Big\{\omega \in B_p^c : \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| > \delta\Big\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} , \quad n \ge n_0(\varepsilon,\delta).$$

[For $\varepsilon > 0$ there exist $p \in \mathbb{N}$ and $B_p \in \mathcal{F}$, $P(B_p) \ge 1 - \frac{\varepsilon}{2k}$. From (4.3') it follows that there exists $n_0 = n_0(\varepsilon)$ such that for $n \ge n_0$

$$\int_{B_p} \Big[\sum_{m=0}^{\infty} c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \Big]^{1/2} dP(\omega) < \frac{\varepsilon}{2}$$

Hence,

$$\int_{\Omega_1} \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| dP(\omega) = \int_{B_p} \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| dP(\omega) +$$

$$\int_{B_p^c} \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| dP(\omega) \le$$

$$\le \int_{B_p} \Big[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \Big]^{1/2} dP(\omega) + k \int_{B_p^c} dP(\omega) \le \frac{\varepsilon}{2} + k \cdot \frac{\varepsilon}{2k} = \varepsilon ,$$

for $n \ge n_0$.]

Theorem 4.3. *Let* $\{\xi_n\}$ *be a sequence of g.r.p.-s on* $\mathcal{A}$. *If* $\xi_n \xrightarrow[b]{P} 0\ (\mathcal{A}')$ $[\xi_n \xrightarrow[b]{P} 0\ (\mathcal{A}')]$ *then for every* $\varepsilon > 0$ *there exist a set* $B \in \mathcal{F}$, *with* $P(B) > 1-\varepsilon$, *an integer* $k_0 \in \mathbb{N}_0$, *both independent of* n, *for each* $m \in \Lambda$ *a sequence of random variables* $\{c_{m,n},\ n \in \mathbb{N}_0\}$, *and for every* $k \geq k_0$ *a sequence of functions* $X_{k,n}$ *of* $\Omega \times I$, *such that*

$$\xi_n(\omega,\phi) = \int_I X_k(\omega,t)\,\mathcal{R}^k\phi(t)\,dt + \sum_{m\in\Lambda} c_{m,n}(\omega)(\psi_m,\phi)\ ,\quad \omega \in B,\quad \phi \in \mathcal{A}\ , \tag{4.5}$$

$$\|X_{k,n}(\omega,\cdot)\|_{L^2} < k\ ,\quad \omega \in \Omega \tag{4.6}$$

$$\|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{P} 0\ ,\quad \left[\|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{1} 0\right] \tag{4.7}$$

(4.8) for every $\delta > 0$ $$P\left\{\omega \in B : \left[\sum_{m\in\Lambda} |c_{m,n}(\omega)| > \delta\right\} \to 0,\ n \to \infty$$

$$\left[\int_B \sum_{m\in\Lambda} |c_{m,n}(\omega)|\,dP \to 0,\ n \to \infty\right]$$

Proof. From Theorem 3.2. and the equivalence of (*ii*) and (*ii'*), (4.5) follows where for $n \in \mathbb{N}_0$, $k \geq k_0$

$$X_{k,n}(\omega,t) = \begin{cases} \displaystyle\sum_{m=0}^{\infty} c_{m,n}(\omega)\,\tilde{\lambda}_m^{-k}\,\psi_m(t)\ , & \omega \in B,\quad t \in I \\ 0 & ,\ \omega \notin B,\quad t \in I \end{cases}$$

Thus we have,

$$\|X_{k,n}(\omega,\cdot)\|^2_{L^2} = \begin{cases} \displaystyle\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2\,|\tilde{\lambda}_m|^{-2k}\ , & \omega \in B,\quad t \in I \\ 0 & ,\ \omega \notin B,\quad t \in I \end{cases} < k$$

and, for $\omega \in B$, $\phi \in \mathcal{A}$

$$\|X_{k,n}(\omega,\cdot)\|_{L^2} = \sup_{\|\phi\|_k \leq 1} \|\xi_n(\omega,\phi)\|$$

Thus, for every $\delta > 0$

$$P\left\{\omega \notin B : \|X_{k,n}(\omega,\cdot)\|_{L^2} > \delta\right\}\ ,$$

and therefore

$$P\left\{ \omega \in \Omega : \|X_{k,n}(\omega,\cdot)\|_{L^2} > \delta \right\} =$$

$$= P\left\{ \omega \in B : \|X_{k,n}(\omega,\cdot)\|_{L^2} > \delta \right\} = P\left\{ \omega \in B : \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| > \delta \right\} \le$$

$$\le P\left\{ \omega \in \Omega : \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| > \delta \right\} \to 0, \quad n \to \infty.$$

Thus, $\|X_{k,n}(\omega,\cdot)\|_{L^2} \to 0$.

$$\left[\int_\Omega \|X_{k,n}(\omega,\cdot)\|_{L^2} dP(\omega) = \int_B \|X_{k,n}(\omega,\cdot)\|_{L^2} dP(\omega) = \right.$$

$$\left. = \int_B \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)|\, dP(\omega) \le \int_\Omega \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)|\, dP(\omega) \to 0 \right].$$

Hence (4.7) follows.

For $m \in \Lambda$ put $\phi = \psi_m$ in (4.5) and we get $P\left\{ \omega \in B \mid |c_{m,n}(\omega)| > \delta \right\} \to 0$ $m \in \Lambda$. Since Λ is finite, (4,8) follows.

Theorem 4.4. $\xi_n \xrightarrow[b]{P} 0\ (\mathcal{A}')$ $[\xi_n \xrightarrow[b]{1} 0\ (\mathcal{A}')]$ *if there exist* $k \in \mathbb{N}_0$ *such that for every* $p \in \mathbb{N}$ *there exist* $B_p \in \mathcal{F}$ *with* $P(B_p) \ge 1 - 1/p$ *such that*

(4.5') $\quad \xi_n(\omega,\phi) = \int_I X_k(\omega,t)\, \mathcal{R}^k f(t)\, dt + \sum_{m\in\Lambda} c_{m,n}(\omega)(\psi_m,\phi)\ , \quad \omega \in B_p, \quad \phi \in \mathcal{A}\ ,$

(4.6') $\quad \|X_{k,n}(\omega,\cdot)\|_{L^2} < k\ , \quad \omega \in \Omega$

(4.7') $\quad \|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{P} 0\ ,\ n \to \infty, \quad \left[\|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{1} 0,\ n \to \infty \right].$

(4.8) $\quad$ for every $\delta > 0 \quad P\left\{ \omega \in B_p : \left[\sum_{m\in\Lambda} |c_{m,n}(\omega)| > \delta \right\} \to 0,\ n \to \infty \right.$

$$\left[\int_B \sum_{m\in\Lambda} |c_{m,n}(\omega)|\, dP \to 0,\ n \to \infty \right].$$

The proof is the same as the proof of Theorem 4.2.

Since

$$\|X_{k,n}(\omega,\cdot)\|_{L^2} = \begin{cases} \sup\limits_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| \ , & \omega \in B_p \\ 0 \ , & \omega \notin B_p . \end{cases}$$

Suppose that (•) and (••) are satisfied.

Theorem 4.5. *Let* $\{\xi_n\}$ *be a sequence of g.r.p.-s on* $\mathcal{A}$. *If* $\xi_n \xrightarrow[b]{P} 0\ (\mathcal{A}')$ $[\xi_n \xrightarrow[b]{1} 0\ (\mathcal{A}')]$, *then for every* $\varepsilon > 0$ *there exist a set* $B \in \mathcal{F}$ *with* $P(B) \ge 1-\varepsilon$, *an integer* $k_0 \in \mathbb{N}_0$, *both independent of* n, *for each* $m \in \Lambda$ *a sequence of random variables* $\{c_{m,n}(\omega),\ n \in \mathbb{N}_0\}$, *and for every* $k \ge k_0$ *a sequence of continuous random processes* $X_{k,n}$ *on* $\Omega \times I$, *such that, for* $n \in \mathbb{N}_0$

(4.9) $$\xi_n(\omega,\phi) = \int_I X_{k,n}(\omega,t)\mathcal{R}^{k+p+s}\phi(t)dt + \sum_{m\in\Lambda} c_{m,n}(\omega)(\psi_m,\phi), \quad \omega \in B,\ \phi \in \mathcal{A},$$

where $s \ge s_0$, $p \ge p_0$.

(4.10) $$\|X_{k,n}(\omega,\cdot)\|_{L^2} < k \ , \quad \omega \in \Omega \ ,$$

(4.11) $\{X_{k,n}(\omega,\cdot)\}$ *is equicontinuous on* I, *for* $p > p_0$,

(4.12) *for each* $t \in I$, *and* $k > k_0$ $X_{k,n}(\cdot,t) \xrightarrow{P} 0$, $n \to 0$,

$$\left[X_{k,n}(\cdot,t) \xrightarrow{1} 0,\ n \to \infty\right]$$

(4.13) for every $\delta > 0$ $P\left\{\omega \in B_p : \left|\sum_{m\in\Lambda} |c_{m,n}(\omega)\right| > \delta\right\} \to 0,\ n \to \infty$

$$\left[\int_B \left|\sum_{m\in\Lambda} |c_{m,n}(\omega)\right| dP \to 0,\ n \to \infty\right] .$$

Proof. From Theorem 3.3. and equivalence of (*ii*) and (*ii'*) (4.9) follow, where for $n \in \mathbb{N}_0$, $k \ge k_0$

$$X_{k,n}(\omega,t) = \begin{cases} \sum\limits_{m=0}^{\infty} c_{m,n}(\omega)\ \tilde{\lambda}_m^{-(k+p+s)} \psi_m(t) \ , & \omega \in B, \quad t \in I \\ 0 \ , & \omega \notin B, \quad t \in I \end{cases}$$

(4.10) and (4.13) follow in the same manner as in Theorem 4.3. The proof of (4.11) is given in Theorem 4.3. of [9].

To prove (4.12) it is enough to see that

$$|X_{k,n}(\omega,t)| \le \sum_{m=0}^{\infty} |c_{m,n}(\omega)|\,|\tilde{\lambda}_m|^{-k}\,|\psi_m(t)|\,|\tilde{\lambda}_m|^{-s}|\tilde{\lambda}_m|^{-p}$$

$$\le K^2\, C \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2\, |\tilde{\lambda}_m|^{-2k} \right]^{1/2}, \quad \omega \in B\,,$$

where $\sum_{m=0}^{\infty} |\tilde{\lambda}_m|^{-2p} = C$. Since $X_{k,n}(\omega,t) = 0$, $\omega \notin B$ we have, for every $\delta > 0$, and $t_0 \in I$

$$P\left\{\omega \in \Omega \;\middle|\; |X_{k,n}(\omega,t_0)| > \delta\right\} \le P\left\{\omega \in B : \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k}\right]^{1/2} > \delta\right\} \le$$

$$\le P\left\{\omega \in \Omega : \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| > \delta\right\} \to 0, \quad n \to \infty\,.$$

$$\Big[\int_{\Omega} |X_{k,n}(\omega,t_0)|\,dP(\omega) = \int_{B} |X_{k,n}(\omega,t_0)|\,dP(\omega) = \int_{B}\left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k}\right]^{1/2} dP(\omega) \le$$

$$\le \int_{\Omega} \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)|\; dP(\omega) \to 0, \quad n \to \infty\,.$$

The converse of Theorem 4.5. is not true.

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Rezime

KONVERGENCIJA NIZA UOPŠTENIH SLUČAJNIH PROCESA NA ZEMANIANOVOM PROSTORU $\mathcal{A}$

Definisane su i uporedene različite vrste konvergencija niza uopštenih slučajnih procesa na Zemanianovom prostoru $\mathcal{A}$.

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ON COINCIDENCE POINTS IN CONVEX METRIC SPACES

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ABSTRACT

In this paper a theorem on coincidence points for f, S and T is proved, where f is a multivalued mapping and S and T are singlevalued mappings. The obtained theorem generalizes Theorem 1 from [3].

1. INTRODUCTION

Many authors proved fixed point theorems or theorems on coincidence points in convex metric spaces [1], [2], [3], [4], [6], [7], [9], [10].

Let us recall that a metric space (M,d) is convex if for each $x,y \in M$ with $x \neq y$ there exists $z \in M$, $x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

In [3] we introduced the notion of a weakly commutative pair (f,S) where f is a multivalued and S is a singlevalued mapping in the following way, where $d(a,B) = \inf_{b \in B} d(a,b)$ and $a \in M$, $B \subseteq M$.

Definition 1. *Let (M,d) be a metric space, K a nonempty sub-*

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set of M, CB(M) *the family of all nonempty, closed and bounded subsets of* M, $f : K \to CB(M)$ *and* $S : K \to M$. *The pair* (f,S) *is said to be weakly commutative if and only if for every* $y \in K$ *and* $z \in K$ *such that* $y \in fz$ *and* $Sz \in K$, *the following inequality holds*

$$d(Sy, fSz) \leq d(fz, Sz).$$

For singlevalued mappings f and S the notion of a weakly commutative pair is introduced by Sessa in [8]. There are examples of mappings which are weakly commutative but not commutative.

We shall give the following generalization of the notion of a weakly commutative pair (f,S) where f is a multivalued and S is a singlevalued mapping.

Definition 2. *Let* (M,d) *be a metric space,* K *a nonempty subset of* M, $f : K \to CB(M)$ *and* $S : K \to M$. *The pair* (f,S) *is said to be compatible if for every sequence* $\{x_n\}_{n\in\mathbb{N}}$ *from* K *from the relations*

$$\lim_{n\to\infty} d(Sx_n, fx_n) = 0 \text{ and } Sx_n \in K,\ n \in \mathbb{N}$$

it follows that

$$\lim_{n\to\infty} d(Sy_n, fSx_n) = 0$$

for every sequence $\{y_n\}_{n\in\mathbb{N}}$ *from* K *such that* $y_n \in fx_n$, $n \in \mathbb{N}$.

For singlevalued mappings S and f the notion of the compatibility is introduced by Jungck [5]. It is obvious that a weakly commutative pair (f,S) is also a compatible one. There are examples of compatible pairs which are not weakly commutative.

In [3] the following result is obtained, where H denotes the Hausdorff metric.

Theorem A. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, $S, T : K \to M$ *continuous mappings,* $f : K \to CB(M)$ H-*continuous mapping,* $\partial K \subseteq SK \cap TK$, $fK \cap K \subseteq SK \cap TK$, (f,S) *and* (f,T) *weakly commutative pairs and the following implications hold:*

$$Tx \in \partial K \Rightarrow fx \in K; \quad Sx \in \partial K \Rightarrow fx \in K.$$

If there exists $q \in (0,1)$ *so that*

$$H(fx,fy) \leq qd(Sx,Ty), \quad \textit{for every } x,y \in K$$

then there exists $z \in K$ *so that* $\{Tz,Sz\} \cap fz \neq \emptyset$.

If $S,T : M \to M$ *are continuous then there exists* $z \in K$ *such that* $Tz \in fz$ and $Sz \in fz$.

Remark. If $S : M \to M$, we suppose in Definition 1 that $y \in fz$ implies $d(Sy,fSz) \leq d(fz,Sz)$ for every $z \in K$ such that $Sz \in K$.

In this paper we shall prove a generalization of Theorem A using the notion of a compatible pair and a result from [7] stated here as Theorem B.

In Theorem B, $\mathbb{R}^+$ stands for the nonnegative reals.

Theorem B. *Let* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function such that* $c(t^+) < t$ *for all* $t > 0$ *and* $\sum_n c^n(t)$ *is finite for all* $t > 0$. *Then, there exists a strictly increasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $c(t) < \psi(t) < t$, *for all* $t > 0$ *and* $\sum_n \psi^n(t)$ *is finite for* $t > 0$.

Theorem. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, S, $T : K \to M$ *continuous mappings,* $f : K \to CB(M)$ *H-continuous mapping,* $\partial K \subseteq SK \cap TK$, $fK \cap K \subseteq SK \cap TK$, (f,S) *and* (f,T) *compatible pairs and the following implications holds:*

$$Tx \in \partial K \Rightarrow fx \in K; \quad Sx \in \partial K \Rightarrow fx \in K.$$

If there exists an increasing function $c : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $c(t^+) < t$, *for all* $t > 0$ *and* $\sum_n c^n(t)$ *is finite, for all* $t > 0$ *so that*

$$H(fx,fy) \leq c(d(Sx,Ty)), \quad \textit{for every } x,y \in K,$$

then there exists $z \in K$ *so that*

$$\{Tz,Sz\} \cap fz \neq \emptyset.$$

If $S, T : M \to M$ *are continuous then there exists* $z \in K$ *so that* $Tz \in fz$, $Sz \in fz$.

Proof. As in [3] let $x \in \partial K$ and $p_0 \in K$ such that $x = Tp_0$. From $Tp_0 \in \partial K$ it follows that $fp_0 \in K \cap fK \subseteq SK$. Hence, there exists $p_1 \in K$ such that $Sp_1 \in fp_0 \subseteq K$ and let $Sp_1 = p_1'$. Further

$$d(p_1', fp_1) \le H(fp_0, fp_1) \le c(d(Sp_1, Tp_0))$$

and if $d(Sp_1, Tp_0) > 0$ from Theorem B we conclude that

$$d(p_1', fp_1) < \psi(d(Sp_1, Tp_0)).$$

So, there exists $p_2' \in fp_1$ such that

$$(1) \qquad d(p_1', p_2') \le \psi(d(Sp_1, Tp_0)).$$

Suppose that $d(Sp_1, Tp_0) = 0$. Then $c(d(Sp_1, Tp_0)) = H(fp_0, fp_1) = 0$ and if we take that $p_2' = p_1'$ we obtain that (1) holds. If $p_2' \in K$ then $p_2' \in K \cap fK \subseteq TK$ and so there exists $p_2 \in K$ such that $Tp_2 = p_2'$. If $p_2' \notin K$ then there exists $p_2 \in K$ such that

$$d(Sp_1, Tp_2) + d(Tp_2, p_2') = d(Sp_1, p_2').$$

Then $d(p_2', fp_2) \le H(fp_1, fp_2) \le c(d(Sp_1, Tp_2))$ and if $d(Sp_1, Tp_2) > 0$ it follows that

$$d(p_2', fp_2) < \psi(d(Sp_1, Tp_2))$$

which implies that there exists $p_3' \in fp_2$ such that

$$(2) \qquad d(p_2', p_3') \le \psi(d(Sp_1, Tp_2)).$$

If $d(Sp_1, Tp_2) = 0$ we take that $p_3' = p_2'$ and so (2) holds.

In this way we obtain two sequences $\{p_n\}_{n \in \mathbb{N}}$ and $\{p_n'\}_{n \in \mathbb{N}}$ such that:

1. For every $n \in \mathbb{N}$: $p_n \in fp_{n-1}$.
2. For every $n \in \mathbb{N}$: $p_{2n}' \in K \Rightarrow p_{2n}' = Tp_{2n}$;

   $p_{2n}' \notin K \Rightarrow Tp_{2n} \in \partial K$ and

$$(3) \qquad d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, p_{2n}') = d(Sp_{2n-1}, p_{2n}').$$

3. For every $n \in \mathbb{N}$: $p'_{2n+1} \in K \Rightarrow p'_{2n+1} = Sp_{2n+1}$;

$p'_{2n+1} \notin K \Rightarrow Sp_{2n+1} \in \partial K$ and

$$d(Tp_{2n}, Sp_{2n+1}) + d(Sp_{2n+1}, p'_{2n+1}) = d(Tp_{2n}, p'_{2n+1}). \tag{4}$$

4. For every $n \in \mathbb{N}$:

$$d(p'_{2n}, p'_{2n+1}) \le \psi(d(Sp_{2n-1}, Tp_{2n})),$$

$$d(p'_{2n+1}, p'_{2n+2}) \le \psi(d(Sp_{2n+1}, Tp_{2n})).$$

Let P_0, P_1, Q_0 and Q_1 be defined by

$$P_0 = \{p_{2n},\ n \in \mathbb{N} \text{ and } p'_{2n} = Tp_{2n}\},$$

$$P_1 = \{p_{2n},\ n \in \mathbb{N} \text{ and } p'_{2n} \ne Tp_{2n}\},$$

$$Q_0 = \{p_{2n+1},\ n \in \mathbb{N} \text{ and } p'_{2n+1} = Sp_{2n+1}\},$$

$$Q_1 = \{p_{2n+1},\ n \in \mathbb{N} \text{ and } p'_{2n+1} \ne Sp_{2n+1}\}.$$

It is easy to prove that

$$(p_{2n}, p_{2n+1}) \notin P_1 \times Q_1, \quad (p_{2n-1}, p_{2n}) \notin Q_1 \times P_1.$$

If $x_{2n} = Tp_{2n}$ and $x_{2n+1} = Sp_{2n+1}$, $n \in \mathbb{N}$ we shall prove that

$$d(x_n, x_{n+1}) \le \begin{cases} \psi(d(x_{n-1}, x_n)), & p'_n \in K \\ \psi(d(x_{n-2}, x_{n-1})), & n \ge 2,\ p'_n \notin K. \end{cases} \tag{5}$$

1. $(p_{2n}, p_{2n+1}) \in P_0 \times Q_0$;

Then we have that

$$d(Tp_{2n}, Sp_{2n+1}) = d(p'_{2n}, p'_{2n+1}) \le \psi(d(Sp_{2n-1}, Tp_{2n}))$$

which means that

$$d(x_{2n}, x_{2n+1}) \le \psi(d(x_{2n-1}, x_{2n})).$$

2. $(p_{2n}, p_{2n+1}) \in P_0 \times Q_1$;

Then from (4) we have that

$$d(Tp_{2n}, Sp_{2n+1}) \le d(Tp_{2n}, p'_{2n+1}) = d(p'_{2n}, p'_{2n+1}) \le \psi(d(Sp_{2n-1}, Tp_{2n}))$$

as in case 1.

3. $(p_{2n}, p_{2n+1}) \in P_1 \times Q_0$;

Then

$$d(Tp_{2n}, Sp_{2n+1}) \le d(Tp_{2n}, p'_{2n}) + d(p'_{2n}, p'_{2n+1}) \le$$

$$\le d(Tp_{2n}, p'_{2n}) + \psi(d(Sp_{2n-1}, Tp_{2n})) \le d(Tp_{2n}, p'_{2n}) +$$

$$+ d(Sp_{2n-1}, Tp_{2n})$$

and from (3) we obtain that

$$d(Tp_{2n}, Sp_{2n+1}) \le d(Sp_{2n-1}, p'_{2n}).$$

Since $p_{2n+1} \in Q_0$ we have $p'_{2n-1} = Sp_{2n-1}$. This implies that

$$d(Tp_{2n}, Sp_{2n+1}) \le d(p'_{2n-1}, p'_{2n}) \le \psi(d(Sp_{2n-1}, Tp_{2n-2}))$$

and so

$$d(x_{2n}, x_{2n+1}) \le \psi(d(x_{2n-1}, x_{2n-2})).$$

4. $(p_{2n-1}, p_{2n}) \in Q_1 \times P_0$;

Then

$$d(Sp_{2n-1}, Tp_{2n}) \le d(Sp_{2n-1}, p'_{2n-1}) + d(p'_{2n-1}, Tp_{2n}) =$$

$$= d(Sp_{2n-1}, p'_{2n-1}) + d(p'_{2n-1}, p'_{2n}) \le$$

$$\le d(Sp_{2n-1}, p'_{2n-1}) + \psi(d(Sp_{2n-1}, Tp_{2n-2})) \le$$

$$\le d(Sp_{2n-1}, p'_{2n-1}) + d(Sp_{2n-1}, Tp_{2n-2})$$

and since $p_{2n-1} \in Q_1$ we obtain that $d(Sp_{2n-1}, Tp_{2n}) \le d(Tp_{2n-2}, p'_{2n-1})$.

From $p_{2n-1} \in Q_1$ it follows that $p_{2n-2} \in P_0$ and so $Tp_{2n-2} = p'_{2n-2}$. Hence

$$d(Sp_{2n-1}, Tp_{2n}) \le d(p'_{2n-2}, p'_{2n-1}) \le \psi(d(Sp_{2n-3}, Tp_{2n-2}))$$

and so

$$d(x_{2n-1},x_{2n}) \leq \psi(d(x_{2n-3},x_{2n-2})).$$

Using 1., 2., 3. and 4. we conclude that (5) is proved. It can be proved that (5) implies that for every $n \in \mathbb{N}$

$$d(x_n,x_{n+1}) \leq \psi^{k(n)}(d(x_0,x_1))$$

where

$$k(n) = \begin{cases} 1, & n = 1 \\ [n/2], & n \geq 2. \end{cases}$$

Since $\sum_n \psi^n(t)$ is finite for $t > 0$ we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and so there exists $z \in K$ such that

$$z = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}.$$

As in [3], suppose that there exists a subsequence $\{p_{2n_k}\}_{k\in\mathbb{N}}$ such that $p_{2n_k} \in P_0$ for every $k \in \mathbb{N}$ which means that $Tp_{2n_k} \in fp_{2n_k-1}$, $k \in \mathbb{N}$. We shall prove that $Sz \in fz$ using the compatibility of (f,S). Since $Tp_{2n_k} \in fp_{2n_k-1} \cap K$, $Sp_{2n_k-1} \in K$ and

$$d(fp_{2n_k-1},Sp_{2n_k-1}) \leq d(Tp_{2n_k},Sp_{2n_k-1})$$

we obtain that $\lim_{k\to\infty} d(fp_{2n_k-1},Sp_{2n_k-1}) = 0$ and from the compatibility of (f,S) it follows that $\lim_{k\to\infty} d(STp_{2n_k},fSp_{2n_k-1}) = 0$. As in [3] it follows that $Sz \in fz$ and so $\{Tz,Sz\} \cap fz \neq \emptyset$. The rest of the proof is similar to that in [3].

Corollary [7]. *Let (X,d) be a complete convex metric space, K a nonempty, closed subset of X and $S : K \to CB(X)$ be such that $Sx \subseteq K$ for every $x \in \partial K$ and*

$$H(Sx,Sy) \leq c(d(x,y)), \text{ for all } x,y \in K$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ is as in the Theorem. Then S has a fixed point.

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REZIME

O TAČKAMA KOINCIDENCIJE U KONVEKSNIM METRIČKIM PROSTORIMA

U radu je dokazana teorema o tačkama koincidencije za f, S i T, gde je f višeznačno preslikavanje a S i T jednoznačna preslikavanja. Dobijena teorema uopštava teoremu 1 iz rada [3].

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